110.302 Lecture 2: Wed, 9/2/15 I

Compare the following:

ⓐ $\frac{dy}{dx} = x - e^{x/2}$ ⓑ $\frac{dp}{dt} = \frac{p}{2} - 450$

① Both are ~~xxx~~ First order ODEs.

② Both are linear (can you ~~xxxx~~ see this?)

③ ⓐ is of the form $y' = f(x)$ ← independent variable on RHS

and is simply a calculus problem

(think: Find $\int f(x)\,dx$) ~~xxxx~~.

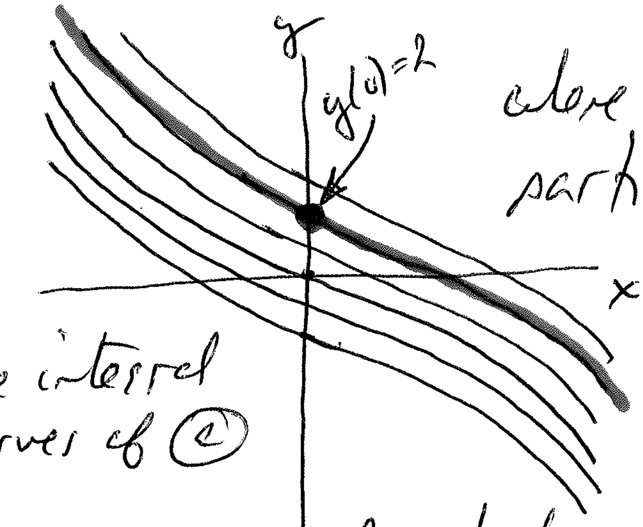
Here the RHS is only a fn of the independent variable.

- Integrate both sides with respect to x to get

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

- This formula is called a <u>general solution</u> to ⓐ as it is an expression that specifies <u>all</u> possible solutions.

- A particular solution to (a) involves choosing a value for the constant c.
- Graphs of solutions are called integral curves (why?)



where each is a particular solution

Some integral curves of (a)

- We also call the general solution to (a) a 1-parameter family of solutions.

With the addition of a single pt in the xy-plane (ex. $y(0)=2$), called an initial value, we can "choose" a particular solution from the family.

With this point, there is now only 1 solution to the problem:

$$y(x) = \frac{x^2}{2} - 2e^{x/2} + C$$

$$y(0) = \frac{0^2}{2} - 2e^{0/2} + C = 2$$

$$= 0 - 2 + C = 2 \implies C = 4$$

Hence the solution to the Initial Value Problem (IVP) (an ODE with initial values)

$$\underbrace{y' = x - e^{x/2},\ y(0) = 2}_{\text{IVP}} \quad \text{is} \quad y(x) = \frac{x^2}{2} - 2e^{x/2} + 4.$$

See the red curve above.

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④ (B) is not of the form $y' = f(x)$.

Rather, it is of the form $y' = f(y)$ ← dependent variable on RHS.

(in this case $p' = f(p)$)

Note: This is harder to solve but easier to study!

Def if in an ODE the independent variable is NOT explicitly present, the ODE is called autonomous.

Let's solve ⑥: there are many ways. We choose something like what we will eventually call "separation of variables".

First, recall from Calculus I,

(I) For any diff. $p(t)$, the functions $p(t)$ and $p(t) - 900$ have the same derivative: $p'(t)$.

(II) $\frac{d}{dt}\left[\ln|f(t)|\right] = \frac{f'(t)}{f(t)}$ for $f(t)$ differentiable and $f(t) \neq 0$.

(chain Rule)

Hence we can rewrite $p' = \frac{p}{2} - 450 = \frac{p-900}{2}$.

as $\frac{p'}{p-900} = \frac{1}{2}$. Why is this useful?

It is useful since the LHS of $\frac{p'}{p-900} = \frac{1}{2}$

looks like the derivative

$$\frac{d}{dt}\left[\ln|p(t)-900|\right] = \frac{1}{2}.$$

Integrate both sides as functions of t to get

$$\ln|p(t)-900| = \frac{t}{2}+C$$

and exponentiate to get

$$|p(t)-900| = e^{t/2+C} = e^{t/2}e^{C}$$ make sense?

If we can "solve" this for $p(t)$ we are done since we would have an expression for the $p(t)$ that solves (6).

Q: Given $|p(t)-900| = e^{t/2}e^{C}$, can $p(t)=900$?

Why or why not?

The ODE (1) has 3 types of solutions:

(1) $p(t) > 900$ always $\Rightarrow p(t) - 900 = Ke^{t/2}$

or $p(t) = 900 + Ke^{t/2}$, where $K = e^c > 0$.

(2) $p(t) < 900$ always $\Rightarrow -(p(t) - 900) = Ke^{t/2}$

or $p(t) = 900 + \underbrace{(-K)}_{\text{negative constant}} e^{t/2}$, where $K = e^c > 0$.

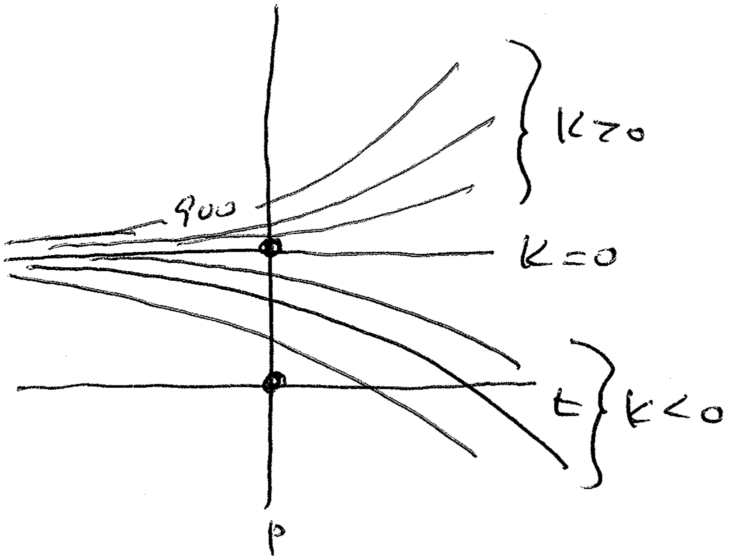
(3) $p(t) = 900$ for all $t \in \mathbb{R}$. It works in the ODE and can also be written

$$p(t) = 900 + Ke^{t/2}, \quad K = 0.$$

This last solution is called a <u>singular solution</u> which sometimes become hidden due to the method employed to find the other solutions.

Note: Our method included a divide by $p - 900$ term which implied we discounted that possibility. We need to account for it.

Conclusion $p(t) = 900 + Ke^{t/2}$, $K \in \mathbb{R}$
is the general solution to (b)



Here the singular solution $p(t) = 900 \ \forall t$ is called an equilibrium solution (or a steady-state soln).

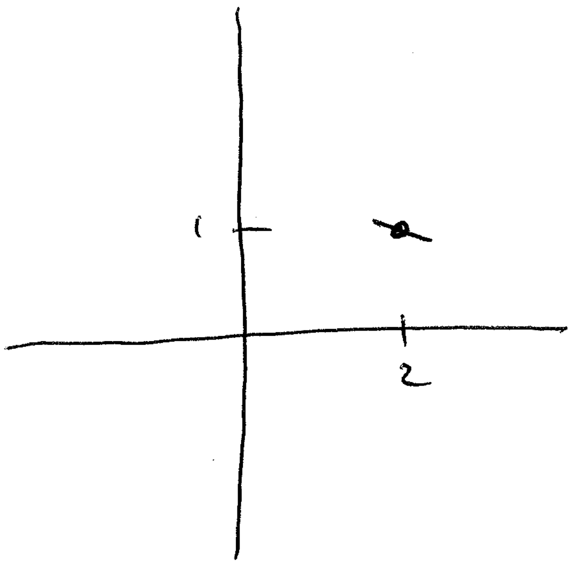
Equilibrium solutions will be very important to us in the future.

Notice in both (a) and (b) above

$\frac{dy}{dx}$ = something involving x, y $\qquad$ $\frac{dp}{dt}$ = something involving t, p

This is useful since solutions "live" in the xy-plane or the tp-plane, any solution curve will have its tangent line at (x,y) with slope given by simply evaluating the RHS at that point.

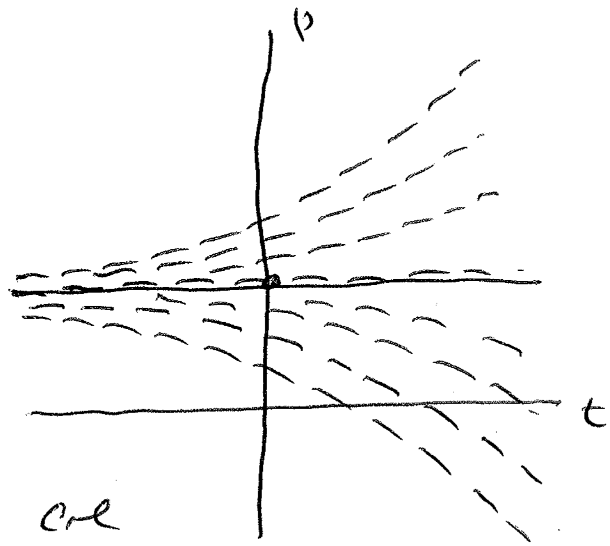
ex. For $y' = x - e^{x/2}$, choose $(x_0, y_0) = (2,1)$.



Then $\left.\dfrac{dy}{dx}\right|_{\substack{x=2\\ y=1}} = 2 - e^{2/2} \cong -.718$

The solution curve passing through $(2,1)$ will have slope $\sim -.718$ there.

Without solving the ODE, we can take a grid of pts in the xy-plane and evaluate these little slope lines. The result is called a slope field:



like blades of grass in a stream with a strong current. These line segments are tangent to all solution curves.

We will use these as a method of study over time.