Lecture 1: August 31, 2015

Start with $y = f(x)$, some unknown functional relation between 2 variables, where

x - independent variable

y - dependent variable

Such an equation is (or sets of them are) called a <u>mathematical</u> <u>model</u> when the variables represent measurable quantities in some application (usually set up to study some unknown entity y based on its relationship to something controllable x).

If $y = f(x)$ is known, then we can simply study its properties (using calculus).

Often, though, we do not know $y = f(x)$, but we do have information about some properties, like derivatives, for example.

<u>examples</u>

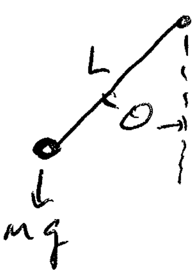
(I) $\frac{dy}{dx} = ky, \quad k \in \mathbb{R}.$

(II) $F = ma$ (Newton's 2^{nd} Law of motion).

(III) $f'(x) = x - e^{x/2}$ (restatement of: Find $\int (x - e^{x/2})\,dx$).

(IV)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



The <u>Pendulum</u>.

The above are mathematical models where the actual function is only known implicitly....

<u>Def</u> An <u>ordinary</u> <u>differential</u> <u>equation</u> (ODE) is an equation involving an unknown function between 2 entities and some of its derivatives.

Note: "Ordinary" means that the unknown function is a function of one independent variable.

ex. The Heat Equation (in 3-space)

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} + \frac{\partial u^2}{\partial z^2}\right)$$

is a partial diff. eqn since u is a fnc of more than 1- indep. variable.

Def. The order of an ODE is the same as the order of the highest derivative that appears in the equation:

ex I and II are first order ODEs.
III and IV are 2nd order ODEs (do you see this?)

Def The general form of an nth order ODE is

$$F(x, y, y', \cdots, y^{(n)}) = 0 \qquad (*)$$

where x - indep var (time?)
y - dep var (unknown fnc $y = f(x)$)
$y^{(i)}$ is the ith derivative of $y = f(x)$
F is some expression in $x, y, y', \cdots, y^{(n)}$

Note: <u>Sometimes</u>, we can solve for the highest derivative:

$$(**) \qquad y^{(n)} = h(x, y, y', \cdots, y^{(n-1)}),$$

But not always:

ex. $y^{(5)} + \sin y^{(5)} = y^{(3)}$ cannot be written like $(**)$. But for $(*)$, $F = y^{(5)} + \sin y^{(5)} - y^{(3)}$.

<u>Def</u>. A function $f(\underbrace{x_1, \cdots, x_n}_{\vec{x}}, y_1, \cdots, y_m)$ is <u>linear</u> in the variables $y_1, \cdots, y_m$ if

$$f(x_1, \cdots, x_n, y_1, \cdots, y_m) = a_0(\vec{x}) + \sum_{i=1}^{m} a_i(\vec{x}) y_i$$

where the $a_i(x)$, $i = 0, \cdots, m$ are arbitrary.

<u>Notes</u> 1 For an ODE to be <u>linear</u>, it must be linear in $y, y', \cdots, y^{(n)}$, and can be written as

$$a_n(\vec{x}) y^{(n)} + \cdots + a_1(\vec{x}) y' + a_0(\vec{x}) y = g(x).$$

examples

Ⓐ $(\sin x)y' + (\ln x)y = \tan e^x$ is a linear 1^{st} order ODE.

Ⓑ $y'' + xy' + \sin y = 0$ is NOT linear

Ⓒ $y''y + y' = 0$ is Not linear.

Suppose $y' = f(t,y)$ is a 1^{st} order linear ODE (written like (**)).

Then there exist functions $p(t)$, $q(t)$ so that $f(t,y) = -p(t)y + q(t)$, and the ODE can be written as

$$y' + p(t)y = q(t)$$

This form will be very important to understanding how to study this type of ODE.

ex. Given $(\sin x)y' + (\ln x)y = \tan e^x$, identify $p(t)$.

Solution: Divide by $\sin x$ to get,

$$y' + \left(\frac{\ln x}{\sin x}\right)y = \frac{\tan e^x}{\sin x}. \quad p(t) = \frac{\ln x}{\sin x}.$$

Def. A <u>solution</u> to $(*)$ on $I=(a,b)\subset\mathbb{R}$ is any function $y=f(x)$ that satisfies the equation $(*)$.

<u>ex.</u> $\frac{dp}{dt}=\frac{p}{2}-450$ is solved by $p(t)=900+ce^{t/2}$ $\forall c\in\mathbb{R}$.

- How do we know? Try it.
- How did we find it? <u>keep listening</u>.
- What to make of the parameter c?

<u>ex.</u> Sometimes a solution is only known <u>implicitly</u>:

Show $x^2+y^2-5=0$ solves $\frac{dy}{dx}=-\frac{x}{y}$.

Here only locally can we solve for $y(x)$.

<u>ex.</u> Solve $x''(t)+\frac{k}{m}x(t)=0$.

<u>ex</u> Solve $y'(x)=x-e^{x/2}$