PROBLEM 4.4.3 FROM THE TEXT

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Solve $y''' - 2y'' - y' + 2y = e^{4t}$ on $\mathbb{R}$.

Strategy. We solve this linear nonhomogeneous third-order ODE with constant coefficients in two ways since we can; first we find a fundamental set of solutions for the homogeneous ODE using the characteristic equation since the coefficients are constant. Then, we utilize the Method of Undetermined Coefficients (UD) to find a particular solution since the forcing function is just an exponential. Then we again solve for the particular nonhomogeneous solution using the Variation of Parameters (VoP) Method.

Solution. First we solve the homogeneous, third-order, linear, constant coefficient ODE $y''' - 2y'' - y' + 2y = 0$. This ODE has characteristic equation $r^3 - 2r^2 - r + 2 = 0$. One way to find possible solutions is to simply try a few low magnitude integers. We see readily that $r = 2$ is a solution. Then we divide the factor $r - 2$ into the polynomial $r^3 - 2r^2 - r + 2$ via long division: It is hard to draw this, but the result is $r^3 - 2r^2 - r + 2 = (r-2)(r^2-1)$. Hence the solutions to the characteristic equation are $r = 1$, -1, and 2. Hence a fundamental set of solutions to the homogeneous ODE is

$$c_1y_1(t) + c_2y_2(t) + c_3y_3(t) = c_1e^t + c_2e^{-t} + c_3e^{2t}.$$

Now, using UD, we assume that a particular solution to the nonhomogeneous ODE has the same form as the forcing function, up to an unknown constant, so $Y(t) = Ae^{4t}$. Then

$$\begin{aligned} Y'''(t) - 2Y''(t) - Y'(t) + 2Y(t) &= e^{4t} \\ (64Ae^4t) - 2(16Ae^{4t}) - (4Ae^{4t}) + 2(Ae^{4t}) &= e^{4t} \\ (64 - 32 - 4 + 2)\, Ae^{4t} &= e^{4t} \end{aligned}$$

yields $A = \frac{1}{30}$. Hence our particular solutions is

$$Y(t) = \frac{1}{30}e^{4t}.$$

Now re-solving using VoP yields a solution guess of $Y(t) = u_1(t)e^t + u_2(t)e^{-t} + u_3(t)e^{2t}$. Running this and its derivatives through the ODE (and making assumptions along the way to simplify), we get the 3×3 system

$$u_1'e^t + u_2'e^{-t} + u_3'e^{2t} = 0 \tag{1}$$

$$u_1'e^t - u_2'e^{-t} + 2u_3'e^{2t} = 0 \tag{2}$$

$$u_1'e^t + u_2'e^{-t} + 4u_3'e^{2t} = e^{4t}. \tag{3}$$

To solve this, we again employ some standard ways of creating equations with the same solutions that have fewer variables. First, add Equation 1 to Equation 2 to get

$$2u_1'e^t + 3u_3'e^{2t} = 0. \tag{4}$$

Then add Equation 2 to Equation 3 to get

$$2u_1'e^t + 6u_3'e^{2t} = e^{4t}. \tag{5}$$

Then, subtract Equation 5 from Equation 4 and get

$$-3u_3'e^{2t} = -e^{4t}.$$

Thus, $u_3' = \frac{1}{3}e^{2t}$ so that $u_3(t) = \frac{1}{6}e^{2t}$.

Take this result and substitute back into Equation 4 yields $2u_1'e^t + 3\left(\frac{1}{3}e^{2t}\right)e^{2t} = 0$, so that $u_1' = \frac{1}{2}\left(-e^{4t}e^{-t}\right) = -\frac{1}{2}e^{3t}$. Thus $u_1(t) = -\frac{1}{6}e^{3t}$.

Now take both of these expressions for $u_1(t)$ and $u_3(t)$ and substitute them back into Equation 1 to get

$$\left(-\frac{1}{2}e^{3t}\right)e^t + u_2'e^{-t} + \left(\frac{1}{3}e^{2t}\right)e^{2t} = 0.$$

Then $u_2' = \frac{1}{6}(e^{4t}e^t) = \frac{1}{6}e^{5t}$. Thus $u_2(t) = \frac{1}{30}e^{5t}$.

Thus our particular solution to the ODE is

$$\begin{aligned} Y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t) \\ &= \left(-\frac{1}{6}e^{3t}\right)e^t + \left(\frac{1}{30}e^{5t}\right)e^{-t} + \left(\frac{1}{6}e^{2t}\right)e^{2t} \\ &= \frac{1}{30}e^{4t}, \end{aligned}$$

as before.