PROBLEM 3.6.6 FROM THE TEXT

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Solve $y'' + 9y = 9\sec^2 3t$ on the interval $0 < t < \frac{\pi}{6}$.

Strategy. We solve the corresponding homogeneous ODE $y'' + 9y = 0$ for the fundamental set of solutions involving $y_1(t)$ and $y_2(t)$. Then we use the method of Variation of Parameters with $Y(x) = u_1(t)y_1(t) + u_2(t)y_2(t)$ to find a particular solution to the original, nonhomogeneous ODE.

Solution. Here, it is readily apparent that a fundamental set of solutions to $y'' + 9y = 0$ is

$$c_1 \cos 3t + c_2 \sin 3t.$$

Indeed, the characteristic equation of this homogeneous, second-order, linear ODE with constant coefficients is $r^2 + 9 = 0$, with solutions $r = \pm 3i$. With the real part 0 and the imaginary part 3, the result follows.

Now, with this assumed form for $Y(t)$, we get the system

$$u_1' \cos 3t + u_2' \sin 3t = 0 \tag{1}$$

$$-3u_1' \sin 3t + 3u_2' \cos 3t = 9\sec^2 3t. \tag{2}$$

Upon multiplying the first equation by $3\sin 3t$ and the second by $\cos 3t$ and adding, we obtain the new equation

$$3u_2' \sin^2 3t + 3u_2' \cos^2 3t = 9\sec^2 3t \cos 3t = 9\sec 3t.$$

Thus $u_2' = 3\sec 3t$ so that

$$u_2(t) = \ln|\sec 3t + \tan 3t| + C,$$

although we will ignore the constant.

Substituting this back into Equation 1, we get

$$u_1' \cos 3t + u_2' \sin 3t = 0 = u_1' \cos 3t + 3\sec 3t \sin 3t, \text{ or } u_1' = -\frac{3\sin 3t}{\cos^2 3t} = -3\tan 3t \sec 3t.$$

Thus, $u_1(t) = -\sec 3t$.

Thus our particular solution to the ODE is

$$\begin{aligned} Y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= -\sec 3t \cos 3t + \ln|\sec 3t + \tan 3t| \sin 3t \\ &= -1 + \ln|\sec 3t + \tan 3t| \sin 3t. \end{aligned}$$

And our general solution to the ODE is

$$y(x) = c_1 \cos 3t + c_2 \sin 3t - 1 + \ln|\sec 3t + \tan 3t| \sin 3t.$$

Now to check whether this particular solution $Y(t)$ is correct as a solution to the nonhomogeneous ODE:

With $Y(t)$ as above, we first calculate the two derivatives:

$$\begin{aligned}
Y(t) &= -1 + \ln|\sec 3t + \tan 3t| \sin 3t, \\
Y'(t) &= 3\sec 3t \sin 3t + 3\ln|\sec 3t + \tan 3t| \cos 3t \\
&= 3\tan 3t + 3\ln|\sec 3t + \tan 3t| \cos 3t, \\
Y''(t) &= 9\sec^2 3t + 9\sec 3t \cos 3t - 9\ln|\sec 3t + \tan 3t| \sin 3t \\
&= 9\sec^2 3t + 9 - 9\ln|\sec 3t + \tan 3t| \sin 3t.
\end{aligned}$$

Now, we place these derivatives back into the original ODE and see

$$\begin{aligned}
Y''(t) + 9Y(t) &= 9\sec^2 3t \\
\big(9\sec^2 3t + 9 - 9\ln|\sec 3t + \tan 3t| \sin 3t\big) + 9\left(-1 + \ln|\sec 3t + \tan 3t| \sin 3t\right) &= 9\sec^2 3t \\
9\sec^2 3t &= 9\sec^2 3t.
\end{aligned}$$