EXAMPLE: PROBLEM 2.6.16 OF THE TEXT

110.302 DIFFERENTIAL EQUATIONS
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Problem. Find the value of the constant $b \in \mathbb{R}$ for which the ODE

$$\left(ye^{2xy} + x\right) + bxe^{2xy}\frac{dy}{dx} = 0 \tag{1}$$

is exact. Then solve, using that value of b.

Strategy. We use the criteria for exactness to calculate the value of b. Then we solve via integration.

Solution. Using the form $M(x,y) + N(x,y)y' = 0$, we identify $M(x,y) = ye^{2xy} + x$, and $N(x,y) = bxe^{2xy}$. The ODE will be exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, or when

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy} = be^{2xy} + 2bxye^{2xy} = b\left(e^{2xy} + 2xye^{2xy}\right) = \frac{\partial N}{\partial x}.$$

Here, the ODE is exact precisely when $b = 1$.

We now solve Equation 1 by calculating a function $\varphi(x,y)$ whose level sets are comprised of solutions to the ODE. First, we assume $M(x,y) = \frac{\partial \varphi}{\partial x}(x,y)$ and integrate with respect to x:

$$\int M\,dx = \int \frac{\partial \varphi}{\partial x}\,dx = \int \left(ye^{2xy} + x\right)\,dx = \frac{1}{2}e^{2xy} + \frac{x^2}{2} + h(y)$$

for some unknown function $h(y)$. Then, knowing that $\varphi(x,y) = \frac{1}{2}e^{2xy} + \frac{x^2}{2} + h(y)$, we calculate $\frac{\partial \varphi}{\partial y}(x,y)$ and identify this with $N(x,y)$:

$$\frac{\partial \varphi}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\frac{1}{2}e^{2xy} + \frac{x^2}{2} + h(y)\right) = xe^{2xy} + h'(y).$$

This must be equal to $N(x,y) = xe^{2xy}$, so that $h'(y) = 0$ or $h(y) = \text{constant}$.

Hence by Theorem, $\varphi(x,y) = \frac{1}{2}e^{2xy} + \frac{x^2}{2} = K$ is the general (implicit) solution to Equation 1.

As a first check, we can verify that this equation in x and y does solve the ODE: For each value of K, $\varphi(x,y) = K$ is the K-level set of the function. This automatically implies that one can view y as an implicit function of x, and hence we can differentiate φ with respect to x, knowing that along the level sets, this derivative will be 0:

$$\frac{d\varphi}{dx}(x,y(x)) = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = \left(ye^{2xy} + x\right) + xe^{2xy}\frac{dy}{dx} = 0$$

which is just the ODE.

But as a second check, we can also solve along the level sets for y as an explicit function of x:

$$\begin{aligned}\frac{1}{2}e^{2xy} + \frac{x^2}{2} &= K \\ e^{2xy} &= 2K - x^2 \\ 2xy &= \ln(2K - x^2) \\ y &= \frac{\ln(2K - x^2)}{2x}.\end{aligned}$$

Note here that we have left the constant as $2K$ to avoid constantly creating new constants by changing the letters. It is all the same in any case.

Here, for specific values of K, one would normally need to then find valid intervals of x where the solution makes sense. But since we are only looking for the general solution, this will do for now. But then, if we substitute this expression for y into our expression for $\varphi(x, y(x))$, we should find that it truly satisfies the expression. To so this, then,

$$\varphi(x, y(x)) = \frac{1}{2}e^{2x\left(\frac{\ln(2K-x^2)}{2x}\right)} + \frac{x^2}{2} = \frac{1}{2}(2K - x^2) + \frac{x^2}{2} = K.$$

It all works.