EXAMPLE: PROBLEM 2.2.30 OF THE TEXT

110.302 DIFFERENTIAL EQUATIONS
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Problem. For the ODE

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}, \tag{1}$$

do the following:

(a) Show that Equation 1 can be rewritten as

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}. \tag{2}$$

Solution. Here,

$$\frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{\left(\frac{y}{x} - 4\right) x}{\left(1 - \frac{y}{x}\right) x} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

through basic algebraic manipulation. Note that we are implicitly making the assumption that $x \neq 0$ in our analysis. However, since we are simply looking at the structure of the ODE for clues as to how to solve it, this is okay. We may have to separately study the cases for initial values $y(0) = y_0$ later.

(b) With the change in variables $v = \frac{y}{x}$ or $y = xv$, write $\frac{dy}{dx}$ in terms of x, v, and $\frac{dv}{dx}$.

Solution. With the change $y = xv$, we can use the Product and Chain Rules to write

$$\frac{dy}{dx} = \frac{d}{dx}[xv] = v + x\frac{dv}{dx}.$$

(c) Now rewrite the ODE in Equation 1 in terms of x and v.

Solution. Here, the ODE is

$$v + x\frac{dv}{dx} = \frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}} = \frac{v - 4}{1 - v}.$$

Thus, we can manipulate this to get

$$\begin{aligned} x\frac{dv}{dx} &= \frac{v - 4}{1 - v} - v = \frac{v - 4}{1 - v} - v\left(\frac{1 - v}{1 - v}\right) \\ &= \frac{v - 4}{1 - v} - \frac{v - v^2}{1 - v} = \frac{v - 4 - v + v^2}{1 - v} = \frac{v^2 - 4}{1 - v}. \end{aligned}$$

Thus the ODE in x and v is

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{3}$$

(d) Solve the ODE in Equation 3, obtaining v as an implicit function of x.

Solution. To solve the new ODE, we first note that it is separable, and we have

$$\frac{1-v}{(v-2)(v+2)}\frac{dv}{dx} = \frac{1}{x}. \tag{4}$$

Integrating the LHS is easier if we decompose the rational function using a partial fraction decomposition:

$$\begin{aligned}\frac{1-v}{(v-2)(v+2)} &= \frac{A}{v-2} + \frac{B}{v+2} = \frac{A(v+2)}{(v-2)(v+2)} + \frac{B(v-2)}{(v-2)(v+2)} \\ &= \frac{Av + 2A + bv - 2B}{(v-2)(v+2)}\end{aligned}$$

so that $1 = 2A - 2B$ and $-1 = A + B$ provide values for the two constants A and B which satisfy the equation. We get $A = -\frac{1}{4}$ and $B = -\frac{3}{4}$.

Integrate Equation 4 with respect to x to get

$$\begin{aligned}\int \left(\frac{1-v}{(v-2)(v+2)}\right)\frac{dv}{dx}\,dx &= \int \frac{1}{x}\,dx \\ -\frac{1}{4}\int \frac{1}{v-2}\,dv - \frac{3}{4}\int \frac{1}{v+2}\,dv &= \ln|x| + C \\ -\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| &= \ln|x| + C.\end{aligned}$$

With a bit of algebraic manipulation, this becomes

$$\ln|v-2| + 3\ln|v+2| + \ln x^4 = -4C.$$

Combine the logarithmic terms and

$$\ln\left|(v-2)(v+2)^3x^4\right| = -4C.$$

And exponentiate to get

$$(v-2)(v+2)^3 = e^{-4C} = K. \tag{5}$$

This last equation expresses v as a implicit function of x.

(e) Rewrite this implicit solution in terms of x and y as a general solution to Equation 1.

Solution. Rewriting Equation 5 in term of y, we get

$$\begin{aligned}\left(\frac{y}{x} - 2\right)\left(\frac{y}{x} + 2\right)^3 x^4 &= K \\ (y-2x)\frac{1}{x}(y+2x)^3\frac{1}{x^3}x^4 &= K \\ (y-2x)(y+2x)^3 &= K.\end{aligned}$$

(f) We will leave the slope field pictures to you.