EXAMPLE: ALMOST LINEAR SYSTEMS: THE PENDULUM

110.302 DIFFERENTIAL EQUATIONS
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Consider the pendulum: a mass m at the end of a rigid, massless rod of length L, fixed at the other end and allowed to swing freely. The movement of the mass can be measured by angular displacement θ from the vertical, with positive displacement to the right. The forces acting on the mass are gravity g pointing downward and damping $c \geq 0$ proportional to velocity and pointing against direction of motion. The second-order ODE governing this motion is

$$mL^2\ddot{\theta} = -cL\dot{\theta} - mgL\sin\theta,$$

or $\ddot{\theta} + \frac{c}{mL}\dot{\theta} + \frac{g}{L}\sin\theta = 0$. By consolidating the constants $\gamma = \frac{c}{mL} \geq 0$ and $\omega^2 = \frac{g}{L} > 0$, we will make the analysis easier (you will see why we choose to create a constant ω^2.) This is the form of the ODE that we wrote out the ODE for the pendulum; it is a second-order, nonlinear, autonomous, ODE with constant coefficients, given as

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2\sin\theta = 0.$$

Written as a system, under the variable change $x = \theta$ and $y = \dot{\theta}$, we get

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\gamma y - \omega^2 \sin x. \end{aligned}$$

Under the idea that solving is quite difficult, what can we say about solutions? What kind of qualitative analysis can we do to well-understand the system? For one thing, we can identify and study the type and stability of the equilibria solutions, or fixed points of the system. These are the places where the steady states occur. These are the place where the two derivatives are simultaneously 0.

First,we find them: Given $F(x,y) = y$, and $G(x,y) = -\gamma y - \omega^2 \sin x$, we solve for the system $f(x,y) = 0$ and $g(x,y) = 0$. We get $y = 0$, and once that is set, we get $\sin x = 0$. Thus the set of all fixed points are

$$\left\{(x,0) \in \mathbb{R}^2 \mid x = n\pi, \;\; n \in \mathbb{Z}\right\}.$$

We can say here that this system, near each fixed point, is almost linear, since the two functions F and G have continuous first derivatives everywhere. Hence to analyze the behavior of solutions near these fixed points, we simply create the associated linear system and study the phase portrait of that linear system. Under the right conditions, we can say a lot about the nonlinear system near these fixed points.

To start, let's analyze the fixed point $(x,y) = (0,0)$. Here, the linear system will have matrix

$$A = \begin{bmatrix} \frac{\partial F}{\partial x}(0,0) & \frac{\partial F}{\partial y}(0,0) \\ \frac{\partial G}{\partial x}(0,0) & \frac{\partial G}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos 0 & -\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}.$$

The characteristic equations of A is $r^2 + \gamma r + \omega^2 = 0$, with solutions

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}.$$

We will be able to say that the original system near the fixed point $(0,0)$ will behave like this linear system with the matrix A provided the eigenvalues satisfy the conditions of the Almost Linear Theorem we talked about in class. We consider some cases:

- **Case I:** Suppose $0 < \gamma < 2|\omega|$. Then the eigenvalues are complex conjugates, with real part given by $\lambda = -\frac{\gamma}{2} < 0$. Solutions to the linear system are sinusoidal with decaying amplitude (you should write them out!). Thus, the linear system has a spiral sink at the origin. By the Theorem, the nonlinear system will also have a spiral sink at the origin. Thus all solutions that start with small displacement angle and small velocity, will have solutions that behave like an amplitude decaying sine curve.
- **Case II:** Suppose $\gamma = 2|\omega|$. Here, the discriminant is 0, and the eigenvalues are real and equal (both are $r = -\frac{\gamma}{2}$. Since $\gamma > 0$, this means that the linear system has either a node at the origin or an improper node (figure out which is true!). In both cases, it is a sink. Hence, by the Theorem, the original system will also have a sink here, although we do not know what kind. We would need a bit of additional analysis to determine this.
- **Case I:** Suppose $\gamma > 2|\omega|$. Here, the eigenvalues are both real and distinct. Your turn! What is going on here? What is happening to the pendulum. Is this case even a real possibility? Discuss....

Now, let's analyze the "other" fixed point (I say other because while there are many fixed points in the plane (all of the $(x,y) = (n\pi, 0)$, for $n \in \mathbb{Z}$), there are really only two, right? This other fixed point is at $(x,y) = (\pi, 0)$. Here the matrix A is

$$A = \begin{bmatrix} \frac{\partial F}{\partial x}(\pi,0) & \frac{\partial F}{\partial y}(\pi,0) \\ \frac{\partial G}{\partial x}(\pi,0) & \frac{\partial G}{\partial y}(\pi,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos\pi & -\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}.$$

The eigenvalues here are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2},$$

and are always real and distinct. In fact, study the last expression for r, the result of the quadratic formula. You will see that it must be the case that for any non-zero value for γ, the two eigenvalues will have different signs, one negative and one positive (why is this?) Thus the equilibrium solution for the linear system is a saddle and unstable. By the Theorem, the non-linear equilibrium at $(\pi, 0)$ is also a saddle and unstable.

Now it should not be surprising that the equilibrium found by balancing the pendulum with the weight vertically over the anchor point is unstable. Build a pendulum and try to find this fixed point (I dare you!) What is surprising is that there exist initial conditions (an initial position of the pendulum and an initial velocity), where the forward trajectory is forever rising up to the equilibrium, never turning back by gravity and never reaching or surpassing the equilibrium. The only real way to see it is to conclude it is there by use of something like the Intermediate Value Theorem.

So here is your thought experiment (Mathematicians build stuff in their head. It saves a lot on construction costs!)

Build a pendulum and conduct a series of trials. Choose a fixed initial point and a small initial velocity, and watch what the pendulum does over time. The pendulum will at some point settle down toward the asymptotically stable equilibrium in the vertical-down position. Record the highest point vertically that the weight reaches. Understand that by fixing the initial position but varying the initial velocity, this highest point of the resulting trajectory will vary continuously with respect to this initial velocity. This is a consequence of the fact that the "function" describing the motion is differentiable). Now increase the initial velocity in some increment and repeat the experiment. Again record the highest point of the resulting trajectory. You should notice that weight traveled higher, closer to the unstable equilibrium before turning around and traveling back down. Keep increasing the initial velocity by the same increment and repeating the experiment. At some point, you will get close to the vertical up equilibrium. And with one more increase in initial velocity and one more experiment, your weight will pass through the vertical-up position before again settling down to the stable equilibrium at the bottom. Now, using smaller decrements in initial velocity, back off your last initial velocity and again start repeating the experiment. At some point, your weight will not cross the top at its first turn. Now again reduce the increments at which you are varying your initial velocity and start increasing your initial velocities again. Continue like this until you get bored, tired, both, or simply have something else to do. By the Intermediate Value Theorem, since varying the initial data results in future positions that also vary continuously from each other, we can conclude that for that fixed starting position, there is a unique starting velocity (in that particular direction) that will result in a trajectory that never turns back down, yet never passes the vertical-up equilibrium. Again, you cannot actually physically see it, but this is a way to "mathematically" see it.

And mathematical sight is as real and fundamental as either physical or sensual sight, no?