Differential Equations – Singular Solutions

Consider the first-order separable differential equation: $\frac{dy}{dx} = f(y)g(x)$. (1)

We solve this by calculating the integrals: $\int \frac{dy}{f(y)} = \int g(x)\,dx + C$. (2)

If y_0 is a value for which $f(y_0) = 0$, then $y = y_0$ will be a solution of the above differential equation (1). We call the value y_0 a *critical point* of the differential equation and $y = y_0$ (as a constant function of x) is called *an equilibrium solution* of the differential equation.

If there is *no* value of C in the solution formula (2) which yields the solution $y = y_0$, then the solution $y = y_0$ is called a *singular solution* of the differential equation (1).

The "general solution" of (1) consists of the solution formula (2) together with all singular solutions.

Note: by "general solution", I mean a set of formulae that produces every possible solution.

Example 1: Solve: $\frac{dy}{dx} = (y-3)^2$. (3)

Solution: $\int \frac{dy}{(y-3)^2} = \int dx$. Thus, $\frac{-1}{(y-3)} = x + C$; $y - 3 = \frac{-1}{x+C}$; and

$$y = 3 - \frac{1}{x+C}, \tag{4}$$

where C is an arbitrary constant.

Both sides of the DE (3) are zero when $y = 3$. No value of C in (4) gives $y = 3$ and thus, the solution $y = 3$ is a singular solution.

The general solution of (3) consists of: $y = 3 - \frac{1}{x+C}$ (C is an arbitrary constant) *and* $y = 3$.

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Example 2: Solve: $$\frac{dy}{dx} = y^2 - 4. \tag{5}$$

Solution: $\displaystyle\int \frac{dy}{y^2-4} = \int dx$. Using partial fractions,

$$\int \frac{dy}{y^2-4} = \int \frac{dy}{(y-2)(y+2)} = \int \frac{1}{4}\left[\frac{1}{y-2} + \frac{-1}{y+2}\right] dy = \int dx.$$

Thus, $\displaystyle\int \left[\frac{1}{y-2} + \frac{-1}{y+2}\right] dy = \int 4\,dx$.

Integrating, $\ln(y-2) - \ln(y+2) = 4x + C$.

Taking exponentials, $\displaystyle\frac{y-2}{y+2} = e^{4x} C_1 = s \quad (\text{say})$.

Then,

$$\begin{aligned} y - 2 &= s(y+2) = sy + 2s \\ y - sy &= 2 + 2s \\ y(1-s) &= 2 + 2s \\ y &= \frac{2+2s}{1-s}. \end{aligned}$$

Thus, $$y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}}, \tag{6}$$

where C_1 is an arbitrary constant.

Both sides of the DE (5) are zero when $y = \pm 2$. If we put $C_1 = 0$ in (6), we obtain the solution: $y = 2$. However, no value of C_1 in (6) gives $y = -2$ and thus, the solution $y = -2$ is a singular solution.

The general solution of (5) consists of:

$$y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}} \quad (C_1 \text{ is an arbitrary constant}) \quad \text{and} \quad y = -2.$$