EXAMPLE: ALMOST LINEAR SYSTEMS: COMPETING SPECIES

110.302 DIFFERENTIAL EQUATIONS
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A common biological model for understanding the possible interaction between 2 species in a closed environment that interact only in their competition for food. Not that one tends to eat the other. More like two herbivores both competing for limited food supplies. If two species did not interact at all, their respective population equations would fit the Logistic Model and be uncoupled:

$$\begin{aligned}\dot{x} &= x(\alpha_1 - \beta_1 x)\\ \dot{y} &= y(\alpha_2 - \beta_2 y)\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are positive constants. We can model a simple interaction between these two species by adding in a cross-term, negative in sign (why?) and scaled by yet another parameter. We get:

$$\begin{aligned}\dot{x} &= x(\alpha_1 - \beta_1 x - \gamma_1 y)\\ \dot{y} &= y(\alpha_2 - \beta_2 y - \gamma_2 x)\end{aligned}$$

where now all $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ are again positive constants. What are the effects of these added terms? How may steady-state solutions (long-term behaviors where the populations of the species do not change over time)? Let's look at these models for a few parameter assignments to see. Do not worry so much about just how modelers came up with this idea of simply adding a term. Let's focus on the solutions for now.

Here are two sets of parameter assignments from Section 9.4:

(1) Let $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\alpha_2 = .75$, $\beta_2 = 1$ and $\gamma_2 = .5$. The system is then

$$\begin{aligned}\dot{x} &= x(1 - x - y)\\ \dot{y} &= y(.75 - y - .5x).\end{aligned}$$

(2) Let $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\alpha_2 = .5$, $\beta_2 = .25$ and $\gamma_2 = .75$. The system is now

$$\begin{aligned}\dot{x} &= x(1 - x - y)\\ \dot{y} &= y(.5 - .25y - .75x).\end{aligned}$$

Look at the slope fields for these two systems on the next two pages and try to identify the differences between these two.

Some questions:

Question 1. Where are the critical points in these systems?

(1) Here, we solve the system

$$0 = x(1 - x - y)$$
$$0 = y(.75 - y - .5x).$$

Of course the origin $\mathbf{a} = (0, 0)$ is one solution. But so are $\mathbf{b} = (0, .75)$, $\mathbf{c} = (1, 0)$, and $\mathbf{d} = (.5, .5)$ (Verify that you know how to find these!).

(2) In this case, the system is

$$0 = x(1 - x - y)$$
$$0 = y(.5 - .25y - .75x).$$

Again, we have $\mathbf{e} = (0, 0)$. The others are $\mathbf{f} = (0, 2)$, $\mathbf{g} = (1, 0)$, and $\mathbf{h} = (.5, .5)$. See these?

Question 2. What are the type and stability of each of these equilibria?

- First note that, for any values of the parameters, the functions $F(x, y) = x(\alpha_1 - \beta_1 x - \gamma_1 y)$ and $G(x, y) = y(\alpha_2 - \beta_2 y - \gamma_2 x)$ are simply polynomials in x and y, and hence by the proposition we did in class, as long as the determinant of the matrix

$$A = \begin{bmatrix} \frac{\partial F}{\partial x}(x_0, y_0) & \frac{\partial F}{\partial y}(x_0, y_0) \\ \frac{\partial G}{\partial x}(x_0, y_0) & \frac{\partial G}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} \alpha_1 - 2\beta_1 x_0 - \gamma_1 y_0 & -\gamma_1 x_0 \\ -\gamma_2 y_0 & \alpha_2 - 2\beta_2 y_0 - \gamma_2 x_0 \end{bmatrix},$$

where (x_0, y_0) is a fixed point, is not 0, the system is almost linear at (x_0, y_0).

(1) In our first case, we have

$$A = \begin{bmatrix} 1 - 2x_0 - y_0 & -x_0 \\ -.5y_0 & .75 - 2y_0 - .5x_0 \end{bmatrix},$$

and at the four critical points, we have

$$A_{\mathbf{a}} = \begin{bmatrix} 1 & 0 \\ 0 & .75 \end{bmatrix}, \quad A_{\mathbf{b}} = \begin{bmatrix} .25 & 0 \\ -.375 & -.75 \end{bmatrix}$$
$$A_{\mathbf{c}} = \begin{bmatrix} -1 & -1 \\ 0 & .25 \end{bmatrix}, \quad A_{\mathbf{d}} = \begin{bmatrix} -.5 & -.5 \\ -.25 & 0 \end{bmatrix}.$$

None of these have determinant 0, so the system is almost linear at all of these equilibria. The eigenvalues tell us that the corresponding linear systems have a source at $\mathbf{a}$, saddles at both $\mathbf{b}$ and $\mathbf{c}$, and a sink at $\mathbf{d}$. (Verify this!) By The Stability Theorem we did in class, the nonlinear equilibria will also have these types and their corresponding stability. The ONLY one of these that is stable is the asymptotically stable sink at $\mathbf{a}$.

(2) Contrast these fixed poitns with those at $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$. We play the same game, and we get the matrix

$$A = \begin{bmatrix} 1 - 2x_0 - y_0 & -x_0 \\ -.75y_0 & .5 - .5y_0 - .75x_0 \end{bmatrix}.$$

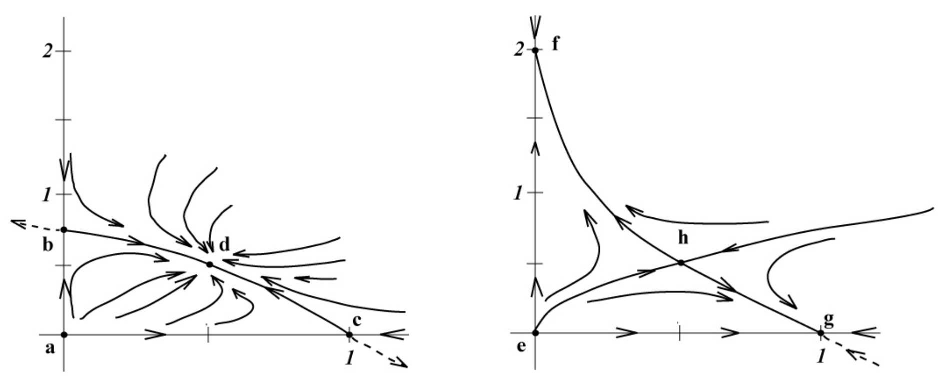
Thus we have the four linear systems given by

$$A_{\mathbf{e}} = \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}, \quad A_{\mathbf{f}} = \begin{bmatrix} -1 & 0 \\ -1.5 & -.5 \end{bmatrix}$$

$$A_{\mathbf{g}} = \begin{bmatrix} -1 & -1 \\ 0 & -.25 \end{bmatrix}, \quad A_{\mathbf{d}} = \begin{bmatrix} -.5 & -.5 \\ -.325 & -.125 \end{bmatrix}.$$

Again, calculate the eigenvalues, and you should find that the linear systems again have an unstable node (a source) at the origin ($\mathbf{e}$), sinks at $\mathbf{f}$ and $\mathbf{g}$, and a saddle at $\mathbf{h}$. All of these are such that the original nolinear equilibria share these characteristics.

Given all of this data, let's place the equilibria and think about how the stability of each would affect the nearby solutions. At the saddles, we would have to find the approximate directions of linear motion. The non-linear saddles will not have linear motion, but they will have something similar; a curve with very specific properties, namely that along one curve, all solution are asymptotic to the equilibrium in forward time. And there will be another curve where all solutions will be asymptotic to the equilibrium in backward time. All other nearby solutions eventually veer away from the equilibrium. The curves of forward and backward asymptotic solutions wind up being tangent at the equilibrium to the directions of linear travel form the linear saddle at that equilibrium. This give an idea of how the non-linear saddle is oriented. We have the two hand drawings below. Your homework now is to go onto JODE, or a similar graphing device, and actually compute slope fields and some numerical solutions to verify that this is more or less correct.



Last question: Suppose we built a system that had sliders for each of the parameters so that we could watch how the phase portrait changed as we alter the parameters continuously. Notice in the two examples above that in both we had $\alpha_1 = \beta_1 = \beta_2 = \gamma_1 = 1$. But in the first example, we had $\alpha_2 = \frac{3}{4}$, $\beta_2 = 1$ and $\gamma_2 = \frac{1}{2}$, and in the second, $\alpha_2 = \frac{1}{2}$, $\beta_2 = \frac{1}{4}$ and $\gamma_2 = \frac{3}{4}$. Imagine if we smoothly slid the values of these parameters from their initial values to their final values, from one diagram to the second. One way to do this is with a single slider: δ, going from 0 to 1. We can use a linear parameterization from any vector $\mathbf{a} \in \mathbb{R}^3$ to any other vector $\mathbf{b} \in \mathbb{R}^3$ to "slide" these parameter values, via $\mathbf{x} = \mathbf{a} + \delta(\mathbf{b} - \mathbf{a})$. Then when $\delta = 0$, $\mathbf{x} = \mathbf{a}$ and when $\delta = 1$, $\mathbf{x} = \mathbf{b}$. Here, we do this with the three parameters above simultaneously:

$$\alpha_2 = \frac{3}{4} + \delta\left(\frac{1}{2} - \frac{3}{4}\right) = \frac{3}{4} - \frac{1}{4}\delta = \frac{3-\delta}{4}$$
$$\beta_2 = 1 + \delta\left(\frac{1}{4} - 1\right) = 1 - \frac{3}{4}\delta = \frac{4-3\delta}{4}$$
$$\gamma_2 = \frac{1}{2} + \delta(\frac{3}{4} - \frac{1}{2}) = \frac{1}{2} + \frac{1}{4}\delta = \frac{2+\delta}{4}.$$

Given this, we can continuously change the phase portrait and look for places where the number, type and/or stability of any of the equilibria change. Given the profound differences between the two phase portraits, we will find something for some intermediate value of rδ.

The Fixed Points, First in this analysis, understand that since we are changing the values of the parameters continuously, the fixed points (equilibria) will either stay where they are or move continuously also. So we can track them. We will do this by rewriting the vector field functions $F(x,y)$ and $G(x,y)$ in terms of δ instead of the parameters: Here, equilibria are the solutions to the equations

$$F(x,y) = 0 = x(1-x-y)$$
$$G(x,y) = y(\alpha_2 - \beta_2 y - \gamma_2 x)$$

become

$$F(x,y) = 0 = x(1-x-y)$$
$$G(x,y) = y\left(\frac{3-\delta}{4} - \frac{4-3\delta}{4}y - \frac{2+\delta}{4}x\right).$$

By inspection, we find:

(1) Whenever $y=0$, $G(x,y)=0$. Hence any equilibria along the x-axis will not depend on δ for position at all. Hence the equilibria at $(0,0)$ and $(1,0)$ do not move for $\delta \in [0,1]$.
(2) For the non-trivial equilibrium along the y-axis, where $x=0$ but $y \neq 0$, $F(x,y)=0$ but $G(x,y)=0$ only when $\alpha_2 - \beta_2 y = 0$, so $y = \frac{\alpha_2}{\beta_2}$. IN terms of δ, there will be a critical point for the system when $x=0$, and
$$y = \frac{\frac{3-\delta}{4}}{\frac{4-3\delta}{4}} = \frac{3-\delta}{4-3\delta}.$$
(3) Lastly, there seems to persist a critical point in the open first quadrant $x,y>0$. This equilibrium will satisfy both
$$\left.\begin{array}{rcl} 1-x-y &=& 0 \\ \alpha_2 - \beta_2 y - \gamma_2 x &=& 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x+y &=& 1 \\ \gamma_2 x + \beta_2 y &=& \alpha_2 \end{array}\right. .$$
Combining these via $y = 1-x$, we get
$$\gamma_2 x + \beta_2(1-x) = \alpha_2, \quad \text{or} \quad x = \frac{\alpha_2 - \beta_2}{\gamma_2 - \beta_2}.$$

In terms of δ, we get

$$x = \frac{\frac{3-\delta}{4} - \frac{4-3\delta}{4}}{\frac{2+\delta}{4} - \frac{4-3\delta}{4}} = \frac{\frac{-1+2\delta}{4}}{\frac{-2+4\delta}{4}} = \frac{1}{2}.$$

Thus $y = 1 - x = \frac{1}{2}$ and we see that the equilibrium strictly in the first quadrant does not move for $\delta \in [0, 1]$ and is at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Thus the four critical points for this model, in terms of δ are

$$(0,0), \quad (1,0), \quad \left(0, \frac{3-\delta}{4-3\delta}\right), \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2}\right).$$

Type and Stability. Now the analysis moves toward a classification of these equilibria for the various values of $\delta \in [0, 1]$. Recall that at any critical point $\mathbf{x}^0 = (x_0, y_0)$ of an almost linear system, we can form the matrix A of an associated linear system, where

$$A = \begin{bmatrix} F_x\big|_{\mathbf{x}^0} & F_y\big|_{\mathbf{x}^0} \\ G_x\big|_{\mathbf{x}^0} & G_y\big|_{\mathbf{x}^0} \end{bmatrix} = \begin{bmatrix} 1 - 2x_0 - y_0 & -x_0 \\ -\gamma_2 y_0 & \alpha_2 - 2\beta_2 y_0 - \gamma_2 x_0 \end{bmatrix}.$$

Here, then

$$A_{(0,0)}(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3-\delta}{4} \end{bmatrix}.$$

Thus, in this case, $r_1 = 1 > 0$ and $r_2 = r_2(\delta) = \frac{3-\delta}{4} > 0$, $\forall \delta \in [0, 1]$. By the Hartman-Grobman Theorem, the equilibrium at the origin is a source for all $\delta \in [0, 1]$.

At the static fixed point at $(1, 0)$, we have

$$A_{(1,0)}(\delta) = \begin{bmatrix} -1 & -1 \\ 0 & \alpha_2 - \gamma_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1-2\delta}{4} \end{bmatrix}.$$

Eigenvalues of $A_{(1,0)}(\delta)$ are immediately available to us since the matrix is upper triangular, so the eigenvalues are the entries on the main diagonal:

$$r_1 = 1, \quad \text{and} \quad r_2 = \frac{1-2\delta}{4}.$$

One can readily show that, via the eigenvector equation, an eigenvector for $r_1 = -1$ is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Along the "other" direction, we have an eigenvalue/eigenvector pair

$$r_2 = \frac{1-2\delta}{4}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{2\delta-5}{4} \end{bmatrix}.$$

The interesting effect is at $\delta = \frac{1}{2}$, where the non-horizontal eigendirection is seen to slow to a stop, creating a curve of equilibria emanating from $(1, 0)$. As δ passes through $\frac{1}{2}$, the eigenvalue r_2 goes from positive to negative, and the saddle bifurcates to a sink, passing through the value where the node is not isolated. This is a planar bifurcation where an unstable node can become stable.

Now, for the critical point $\left(0, \frac{3-\delta}{4-3\delta}\right)$, we get

$$A_{\left(0, \frac{3-\delta}{4-3\delta}\right)}(\delta) = \begin{bmatrix} 1 - y_0 & 0 \\ \gamma_2 y_0 & \alpha_2 - \beta_2 y_0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3-\delta}{4-3\delta} & 0 \\ -\left(\frac{2+\delta}{4}\right)\left(\frac{3-\delta}{4-3\delta}\right) & \frac{3-\delta}{4} - 2\left(\frac{4-3\delta}{4}\right)\left(\frac{3-\delta}{4-3\delta}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1-2\delta}{4-3\delta} & 0 \\ \left(\frac{2+\delta}{4-3\delta}\right)\left(\frac{\delta-3}{4}\right) & \frac{\delta-3}{4} \end{bmatrix}.$$

Here, the eigenvalues are $r_1 = \frac{1-2\delta}{4-3\delta}$, and $r_2 = \frac{\delta-3}{4}$. For the eigenvector $\mathbf{v}_2 = [v_1, v_2]^T$ corresponding to r_2, we have the eigenvector system

$$\left(\frac{1-2\delta}{4-3\delta}\right) v_1 = \left(\frac{\delta-3}{4}\right) v_1$$
$$\left(\frac{2+\delta}{4}\right)\left(\frac{\delta-3}{4}\right) v_1 + \left(\frac{\delta-3}{4}\right) v_2 = \left(\frac{\delta-3}{4}\right) v_2.$$

This system is solve by $v_1 = 0$ and v_2 is anything non-trivial, so that the vector $\mathbf{v}_2$ is along the y-axis $\forall \delta \in [0, 1]$.

For r_1, eigenvector system is

$$\frac{1-2\delta}{4-3\delta} v_1 = \frac{1-2\delta}{4-3\delta} v_1$$
$$\left(\frac{2+\delta}{4}\right)\left(\frac{\delta-3}{4}\right) v_1 + \left(\frac{\delta-3}{4}\right) v_2 = \left(\frac{1-2\delta}{4-3\delta}\right) v_2.$$

While this is a fairly messy calculation, we can boil it down to

$$v_1 = \frac{3\delta^2 - 21\delta + 16}{(2+\delta)(\delta-3)} v_2.$$

Upon inspection, one can readily see that both components will be non-zero for every $\delta \in [0, 1]$, except for at one value: $\delta \sim .87$. At this point, one can show, $r_1 = r_2$, and there is only one eigendirection.

Exercise 1. Establish what is happening for this value of δ at the non-trivial critical point along the vertical axis.

Lastly, for this case, notice again, that one of the eigenvalues $r_1 = 0$, when $\delta = \frac{1}{2}$. This is precisely another instance of a bifurcation from a saddle to a sink, where one of the eigendirections slows down its repellent motion, stops and then reverses direction. Interesting....

And lastly, Let's analyze the stability, type and structure of the phase space at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. We have

$$A_{\left(\frac{1}{2}, \frac{1}{2}\right)}(\delta) = \begin{bmatrix} 1 - 2x_0 - y_0 & -x_0 \\ -\gamma_2 y_0 & \alpha_2 - 2\beta_2 y_0 - \gamma_2 x_0 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{2+\delta}{8} & \frac{3-\delta}{4} - \frac{4-3\delta}{4} - \frac{2+\delta}{8} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{2+\delta}{8} & \frac{-4+3\delta}{8} \end{bmatrix},$$

with eigenvalues

$$r = -\frac{(-8+3\delta) \pm \sqrt{(-8+3\delta)^2 - (32-64\delta)}}{16} = \frac{(8-3\delta) \pm \sqrt{9\delta^2 + 16\delta + 32}}{16}.$$

One can easily see by inspection here that both of the eigenvalues here are real, with one of them remaining negative for all $\delta \in [0,1]$. The other one, however, is negative on $\delta \in \left[0, \frac{1}{2}\right)$, and positive on $\delta \in \left[\frac{1}{2}, 1\right)$, and 0, when $\delta = \frac{1}{2}$. This, again, denotes a bifurcation value for δ, with the equilibrium going from a sink to a saddle.

IN fact, at $\delta = \frac{1}{2}$, we have that strange situation where the three non-trivial critical points all have 0 as an eigenvalue of their linearization. This suggests a curve of critical points in the phase plane. We can actually of directly to the original differential equation to find these:

Let $\delta = \frac{1}{2}$. Then the critical points are all at

$$\begin{aligned}
F(x,y) &= x(1-x-y) = 0 \\
G(x,y) &= y\left(\frac{3-\frac{1}{2}}{4} - \left(\frac{4-\frac{3}{2}}{4}\right)y - \left(\frac{2+\frac{1}{2}}{4}\right)x\right) = 0 \\
&= y\left(\frac{5}{8} - \frac{5}{8}y - \frac{5}{8}x\right) = 0 \\
&= \frac{5}{8}y(1-x-y) = 0.
\end{aligned}$$

At $\delta = \frac{1}{2}$, there will be a line of critical points, ranging within the first quadrant along the line $y = 1 - x$, from the equilibrium at $(1,0)$, through the equilibrium at $\left(\frac{1}{2}, \frac{1}{2}\right)$ to the the fixed point at $(1,0)$. Can you envision this?