APPLICATION WORKSHEET: RESONANCE

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

A car supported by a MacPherson strut (shock absorber system) travels on a bumpy road at a constant velocity ν. The equation modeling the motion of the car is

$$80\ddot{x} + 10000x = 2500\cos\left(\frac{\pi\nu t}{6}\right), \tag{1}$$

where $x(t)$ represents the vertical position of the car's axle relative to its equilibrium position, and the basic units of measurement are feet and feet per second (this is actually just an example of a forced, un-damped harmonic oscillator, if that is any help). The constant numbers above are related to the characteristics of the car and the strut. Note that the coefficient of time t (inside the cosine) in the forcing term on the right hand side is a frequency, which in this case is directly proportional to the velocity ν.

(a) Find the general solution to this non-homogeneous ODE. Note that your answer will have a term in it which is a function of ν.

(b) Determine the value of ν for which the solution is undefined (you should present your final answer in miles per hour, as opposed to feet per second).

(c) For a set of initial values $x(0) = 0$ and $\dot{x}(0) = 0$, graph the solutions for a few values of ν near your answer in part b and not so near. Discuss the differences in these graphs and the importance of the special value of ν in part (b). (Hint: This special value of ν induces what is called resonance in the car).

(d) Write the IVP (the ODE with the initial conditions in part (c)) as a non-homogeneous first order system of ODEs. We will learn how to solve such a system in time.

Solution.

(a) First, notice that ODE in Equation 1 is a second-order linear nonhomogeneous ODE with constant coefficients. To find the general solution, we will first find a set of two linearly independent solutions to the homogeneous version

$$80\ddot{x} + 10000x = 0.$$

The characteristic equation for this ODE is then $80r^2 + 10000 = 0$, which is solved by $r = \pm\sqrt{-125} = \pm 5\sqrt{5}\, i$. Being purely imaginary (and hence complex), we can immediately write out the general solution of this homogeneous, linear constant coefficient second order ODE, and get

$$x(t) = c_1\cos 5\sqrt{5}t + c_2\sin 5\sqrt{5}t.$$

The general solution to the original nonhomogeneous ODE is then

$$x(t) = c_1\cos 5\sqrt{5}t + c_2\sin 5\sqrt{5}t + X(t),$$

where $X(t)$ is any solution to Equation 1. To find a suitable representative for $X(t)$, we can use the Method of Constant Coefficients:

Choose $X(t) = A\cos\left(\frac{\pi\nu t}{6}\right)$. We would normally also include another constant involving the sine function, but since there is no first derivative in the ODE, there will not be a sine component (Stop here for a minute to absorb and accept this idea).

Now since $\ddot{X}(t) = -A\left(\frac{\pi\nu}{6}\right)^2\cos\left(\frac{\pi\nu t}{6}\right)$, we can sub these into the ODE to get

$$\begin{aligned}
80\left(-A\left(\frac{\pi\nu}{6}\right)^2\cos\left(\frac{\pi\nu t}{6}\right)\right) + 10000\left(A\cos\left(\frac{\pi\nu t}{6}\right)\right) &= 2500\cos\left(\frac{\pi\nu t}{6}\right) \\
\left(10000 - 80\left(\frac{\pi\nu}{6}\right)^2\right)A\cos\left(\frac{\pi\nu t}{6}\right) &= 2500\cos\left(\frac{\pi\nu t}{6}\right) \\
A\cos\left(\frac{\pi\nu t}{6}\right) &= \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right) \\
A &= \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}.
\end{aligned}$$

Hence the general solution to Equation 1 is

$$\boxed{x(t) = c_1\cos 5\sqrt{5}t + c_2\sin 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right)}.$$

(b) You can see how the speed of the car would affect the solution. In fact, there is a special value of ν for which this solution is NOT defined. That is, when $10000 - 80\left(\frac{\pi\nu}{6}\right)^2 = 0$. This is solved by

$$\begin{aligned}
10000 &= 80\left(\frac{\pi\nu}{6}\right)^2 \\
\sqrt{\frac{10000}{80}} &= \frac{\pi\nu}{6} \\
\frac{6\sqrt{125}}{\pi} &= \boxed{\nu \cong 21.35 \text{ feet per second} \cong 14.55 \text{ miles per hour}}.
\end{aligned}$$

(c) Here, we find the particular solution corresponding to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. For expediency, let's start with the second one. We calculate:

$$\begin{aligned}
\dot{x}(0) &= \frac{d}{dt}\left[c_1\cos 5\sqrt{5}t + c_2\sin 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right)\right]\Bigg|_{t=0} \\
&= \left(-5\sqrt{5}c_1\sin 5\sqrt{5}t + 5\sqrt{5}c_2\cos 5\sqrt{5}t - \left(\frac{\pi\nu}{6}\right)\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\sin\left(\frac{\pi\nu t}{6}\right)\right)\Bigg|_{t=0} \\
&= 5\sqrt{5}c_2\cos 5\sqrt{5}(0) = 0
\end{aligned}$$

which implies that $c_2 = 0$. Thus the solution is now

$$x(t) = c_1\cos 5\sqrt{5}t + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\cos\left(\frac{\pi\nu t}{6}\right).$$

As for the other initial condition, we get

$$\begin{aligned} x(0) &= c_1 \cos 5\sqrt{5}(0) + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} \cos\left(\frac{\pi\nu(0)}{6}\right) \\ &= c_1 + \frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} = 0 \\ \implies \quad c_1 &= \frac{-2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2} \end{aligned}$$

Thus our particular solution is

$$x(t) = \left(\frac{-2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right) \cos 5\sqrt{5}t + \left(\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right) \cos\left(\frac{\pi\nu t}{6}\right)$$

or

$$\boxed{x(t) = \left(\frac{2500}{10000 - 80\left(\frac{\pi\nu}{6}\right)^2}\right)\left(\cos\left(\frac{\pi\nu t}{6}\right) - \cos 5\sqrt{5}t\right)}.$$

What does this solution look like for various values of ν? First, think about it. As the difference between two cosine functions of different frequencies, there will usually be a periodic pattern that develops. Convince yourself of this. However, as ν approaches the special value which renders the coefficient fraction undefined, the amplitude of the oscillations grow. As long as we are off this special value, we will still see periodic behavior. But with the amplitude getting larger and larger, at some point, the strut will break and the car will become inoperable. We see the graph near this resonant value of ν as a form of constructive interference, the oscillations getting larger and larger until something breaks. This car speed provides the resonant frequency at which the car's vibrations become unbounded. Note here critically, the even for values near the resonant car speed, the vibrations will be periodic, although with amplitudes that get much too large for the strut to endure.

On the next page, I plot some graphs of $x(t)$ for various values of ν.

Note. *Notice the amplitude change in the graphs on the next page as ν gets close to the resonant value, and then again as ν gets large. Watching the vertical axis scaling.*

(d) Written as a system, we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -125x_1 + 2500\cos\left(\frac{\pi\nu t}{6}\right) \end{aligned},$$

or as a matrix ODE:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -125 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2500\cos\left(\frac{\pi\nu t}{6}\right) \end{bmatrix}.$$

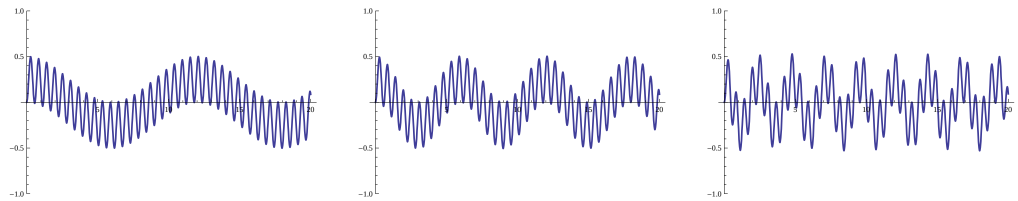

Figure 1. $x(t)$ when $\nu = 1$ (left), when $\nu = 2$ (center), and when $\nu = 5$ (right).

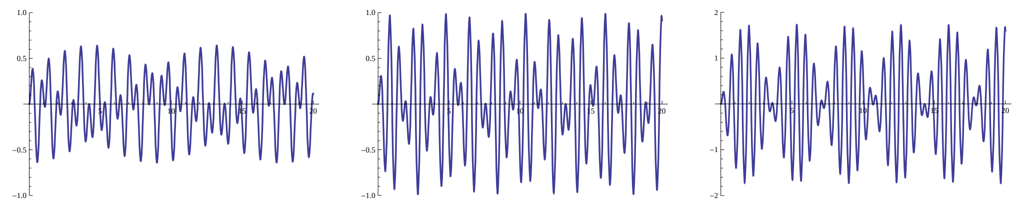

Figure 2. $x(t)$ when $\nu = 10$ (left), when $\nu = 15$ (center), and when $\nu = 18$ (right).

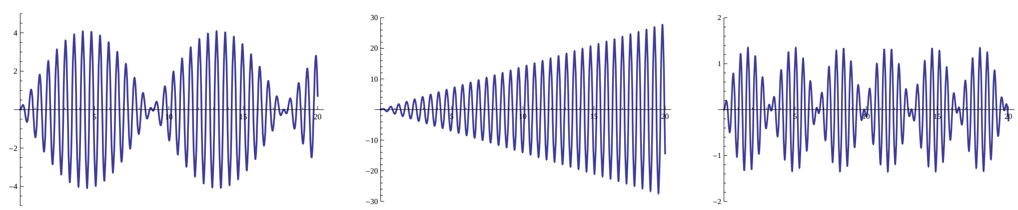

Figure 3. $x(t)$ when $\nu = 20$ (left), when $\nu \cong 21.35$ (center), and when $\nu = 25$ (right).

Figure 4. $x(t)$ when $\nu = 30$ (left), when $\nu = 50$ (center), and when $\nu = 100$ (right).