EXAMPLE: THE WRONSKIAN DETERMINANT OF A SECOND-ORDER, LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

110.302 DIFFERENTIAL EQUATIONS
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Given a second order, linear, homogeneous differential equation

\[ y'' + p(t)y' + q(t)y = 0, \]

where both \( p(t) \) and \( q(t) \) are continuous on some open \( t \)-interval \( I \), and two solutions \( y_1(t) \) and \( y_2(t) \), one can form a fundamental set of solutions as the linear combination of these two

\[ y(t) = c_1y_1(t) + c_2y_2(t) \]

ONLY under the condition that the Wronskian determinant

\[ W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0 \]

for all \( t \in I \). This condition implies that the two differentiable functions \( y_1(t) \) and \( y_2(t) \) are independent; that there does not exist two constants \( k_1, k_2 \in \mathbb{R} \), not both equal to 0, where \( k_1y_1(t) + k_2y_2(t) = 0 \) on \( I \). Then we know that ALL solutions to the ODE will be a linear combination of \( y_1(t) \) and \( y_2(t) \).

We can actually take this further. It turns out that in this situation, and on the interval \( I \), either of two things can happen:

- The Wronskian is always 0 on \( I \) (we say \( W(y_1, y_1) \) is identically 0 on \( I \), or \( W(y_1, y_2) \equiv 0 \) on \( I \), or
- the Wronskian is NEVER 0 on \( I \).

Why? Because of the following:

**Theorem.** If \( y_1(t) \) and \( y_2(t) \) are two solutions to the ODE \( y'' + p(t)y' + q(t)y = 0 \), where \( p(t) \) and \( q(t) \) are continuous on some open \( t \)-interval \( I \), then

\[ W(y_1, y_2)(t) = Ce^{-\int p(t) \, dt} \]

where \( C \) depends on the choice of \( y_1 \) and \( y_2 \), but not on \( t \).

First, some notes:

- If \( y_1(t) \) and \( y_2(t) \) are linearly dependent, then \( C = 0 \).
- If \( y_1(t) \) and \( y_2(t) \) are linearly independent, then \( W(y_1, y_2) \neq 0 \) on ALL of \( I \).
- Linear independence and non-zero Wronskian are the same thing for solutions to these ODEs.
Proof. Since $y_1$ and $y_1$ both solve the ODE, we have
\begin{align*}
y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\
y_2'' + p(t)y_2' + q(t)y_2 &= 0.
\end{align*}

Here, multiply the first equation by $-y_2$ and the second by $y_1$ and then add them together (this will eliminate the coefficient $q(t)$)
\begin{equation*}
\underbrace{(y_1y_2' - y_2y_1')}_{W'(y_1,y_2)} + p(t) \underbrace{(y_1y_2' - y_2y_1')}_{W(y_1,y_2)} = 0,
\end{equation*}

Which leads to a first order differential equation whose variable IS the Wronskian determinant itself (really, the Wronskian is a function of $t$)
\begin{equation*}
W' + p(t)W = 0.
\end{equation*}

This first order ODE is both linear and separable, and by separation of variables, we get
\begin{equation*}
\frac{W'}{W} = -p(t) \implies \ln |W| = - \int p(t) \, dt + K \implies Ce^{-\int p(t) \, dt}.
\end{equation*}

Hence by the notes above just before the proof, either $C = 0$, and the Wronskian is always 0, and the two solutions are linearly dependent, or $C \neq 0$, and the Wronskian is NEVER 0, and the two solutions are linearly independent.