THEORY: SEPARABLE DIFFERENTIAL EQUATIONS

110.302 DIFFERENTIAL EQUATIONS
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A first-order differential equation of the form

$$y' = f(t, y)$$

is called *separable* if the function f can be written

$$f(t, y) = g(t)h(y),$$

for two other functions $g(t)$ and $h(y)$. In other words, if f can be separated into the product of two functions, one ONLY of the independent variable t and the other ONLY of the dependent variable y. This is very useful, as we can then separate the variables into different sides of the equations, and use standard integration to uncover a solution.

Indeed, if $y' = f(t, y) = g(t)h(y)$, then we can divide by the function $h(y)$ on both sides and write

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t).$$

Since we are only looking for an idea to study the ODE, we can do this without concern about the places where $h(y)$ may be zero. Later, when looking for particular solutions, we will have to worry about this. In this last equation, all of the ys are on one side and all of the ts are on the other. Both sides really are functions of t, since the unknown function $y(t)$ is (thus $h(y)$, $\frac{1}{h(y)}$ and y' are functions of t). Since the left-hand side equals the right-hand side, their antiderivatives are also equal, up to a constant. Hence

$$\int \left(\frac{1}{h(y)}\frac{dy}{dt}\right)\, dt = \int g(t)\, dt. \tag{0.1}$$

The right-hand side of this last equation is straightforward, as long as the antiderivative can be found. But the left-hand side looks complicated. It is, but not really.

Caution. *There is an urge to simplify the left-hand side by simply canceling out the two dt terms, thereby making the integral an integral solely in the variable y. It turns out the end result may be okay (as we will see), but the logic is WRONG! The derivative term $\frac{dy}{dt}$ is NOT a fraction. Hence it does not have a denominator. Such an incorrect short-cut may see you through this tough spot, but the lack of real understanding will continue to haunt you as we progress. Instead, use your Calculus I skills to "see" the correct way!*

To see what is going on in this left-hand side, realize that the first term $\frac{1}{h(y)}$ is really a composition of functions. Think of it this way: Write the left-hand side using $u(t)$ instead of y. Here

$$\int \frac{1}{h(u(t))}\frac{du}{dt}\, dt.$$

This integrand is simply a product of functions where one factor is the composition of the functions $\frac{1}{h(t)}$ and $u(t)$, and the other factor is the derivative of the "inside" function

$u(t)$. If we use the standard Substitution Method from Calculus I to make the substitution $y = u(t)$ to the new variable y, we calculate the differential of y in terms of u and get $dy = u'(t)\, dt = \frac{du}{dt}\, dt$, and

$$\int \frac{1}{h(u(t))} \frac{du}{dt}\, dt = \int \frac{1}{h(y)}\, dy.$$

The substitution method has partially untangled the integrand, making the variable of integration now y for this antiderivative. This looks the same as if we simply canceled out the dts from the left-hand integral in Equation 0.1 as in the caution. However, that is not correct!

So, integration step given in Equation 0.1 becomes

$$\int \left(\frac{1}{h(y)} \frac{dy}{dx} \right) dx = \int \frac{1}{h(y)}\, dy = \int g(t)\, dt.$$

Both the middle and right-hand side are STILL equal as antiderivatives of their respective integrands, yet the left-hand side has been altered via the Substitution Method to be more manageable. The rest of the calculation is now straightforward: Find the anti-derivatives, if possible, for each side, and solve for $y(t)$, again, if possible.

Example 0.1. Solve the Ordinary Differential Equation $y' + y^2 \sin x = 0$.

Strategy. This ODE is NOT linear, due to the exponent on the y variable. But it is separable. Here, we separate variables, then integrate to expose an equation involving y and x. Then we attempt to solve for y as an explicit function of x, if possible.

Solution. Placed into the form $y' = g(x)h(y)$, we have

$$y' = -(\sin x) y^2,$$

so that $g(x) = -\sin x$ and $h(y) = y^2$. Separate and then integrate to get

$$\begin{aligned} y' &= -(\sin x) y^2 \\ \frac{1}{y^2} \frac{dy}{dx} &= -\sin x \\ \int \left(\frac{1}{y^2} \frac{dy}{dx} \right) dx &= -\int \sin x\, dx \\ \int \frac{1}{y^2}\, dy &= -\int \sin x\, dx \\ -\frac{1}{y} &= \cos x + C. \end{aligned}$$

This last equation can be "solved" for y as a function of x, by inverting both sides (if two expressions are equal, their reciprocals (where they make sense) will also be equal). Thus

$$\begin{aligned} -\frac{1}{y} &= \cos x + C \\ y(x) &= \frac{-1}{\cos x + C}. \end{aligned}$$

Be very careful where the constant of integration goes here! Thus this last equation is our general solution to the separable first-order ODE.

Is it correct? We check: Given $y(x) = \dfrac{-1}{\cos x + C}$ as above, we calculate

$$y'(x) = \frac{d}{dx}\left[\frac{-1}{\cos x + C}\right] = (-1)\frac{-1}{(\cos x + C)^2}(-\sin x) = \frac{-\sin x}{(\cos x + C)^2}.$$

Substitute both $y(x)$ and $y'(x)$ into the original linear ODE, and we get

$$\begin{aligned} y' + y^2 \sin x &= 0 \\ \frac{-\sin x}{(\cos x + C)^2} + \left(\frac{-1}{\cos x + C}\right)^2 \sin x &= 0 \\ \frac{-\sin x}{(\cos x + C)^2} + \frac{\sin x}{(\cos x + C)^2} &= 0. \end{aligned}$$

Again, it all works.