Name: ______________________________ Section Number: ____________

110.302 Differential Equations
FALL 2012
MIDTERM EXAMINATION D
December 5, 2012

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		20
3		20
4		20
5		25
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:** ____________

PLEASE SHOW ALL WORK, EXPLAIN YOUR REASONS, AND STATE ALL THEOREMS YOU APPEAL TO

Question 1. [15 points] Suppose $y(t)$ is the solution to the IVP $y' = 4t - 4y^2$, $y(0) = 1$. Using the Euler Method with a step size of $\frac{1}{2}$, estimate $y(2)$.

Euler Method: $t_{n+1} = t_n + h$

$y_{n+1} = y_n + h f(t_n, y_n)$

Here $t_0 = 0$, $y_0 = y(t_0) = y(0) = 1$.

$t_1 = \frac{1}{2}$, $y_1 = y(\frac{1}{2}) = y_0 + \frac{1}{2}(4t_0 - 4y_0^2) = 1 + \frac{1}{2}(4(0) - 4(1)^2)$

$= 1 + \frac{1}{2}(-4) = -1.$

$t_2 = 1$, $y_2 = y(1) = y_1 + \frac{1}{2}(4t_1 - 4y_1^2) = -1 + \frac{1}{2}(4(\frac{1}{2}) - 4(-1)^2)$

$= -1 + \frac{1}{2}(2-4) = -1 - 1 = -2.$

$t_3 = \frac{3}{2}$, $y_3 = y(\frac{3}{2}) = y_2 + \frac{1}{2}(4t_2 - 4y_2^2) = -2 + \frac{1}{2}(4(1) - 4(-2)^2)$

$= -2 + \frac{1}{2}(4-16) = -2 - 6 = -8$

$t_4 = 2$, $y_4 = y(2) = y_3 + \frac{1}{2}(4t_3 - 4y_3^2) = -8 + \frac{1}{2}(4(\frac{3}{2}) - 4(-8)^2)$

$= -8 + \frac{1}{2}(6 - 256) = -8 + \frac{1}{2}(250) = -133$

$\boxed{\text{Solution: For the solution } y(t) \text{ to the IVP, } y(2) \approx -133.}$

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Question 2. [20 points] Do the following:

(a) Solve the Initial Value Problem

$$5y'' - 5y' - 10y = 10e^{-t}, \quad y(0) = 1,\ y'(0) = -\frac{2}{3}.$$

(b) Using the Method of Undetermined Coefficients, what would your guess of a particular solutions to the non-homogeneous ODE be if the forcing function (the right-hand-side) of the equation was $10e^{2t}(\cos 3t) + 5t^3$?

(a) Same as other test after mult. by common coefficient factor $\frac{4}{5}$.

(b) Guess here is

$$\boxed{Y(t) = Ae^{2t}\cos 3t + Be^{2t}\sin 3t + Ct^3 + Dt^2 + Et + F.}$$

Question 3. [20 points] For the second-order, linear, ODE

$$2p'' + 4p' + 10p = 0,$$

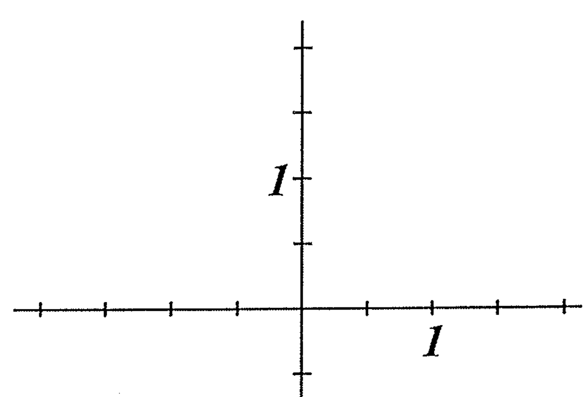
do the following:

(a) Rewrite the equation as a system of two first-order ODEs.

(b) Find the general vector solution to the associated linear homogeneous system.

(c) Carefully draw the phase portrait for this system.

All same as other exam upon multiplication of entire eqn by $\frac{3}{2}$.



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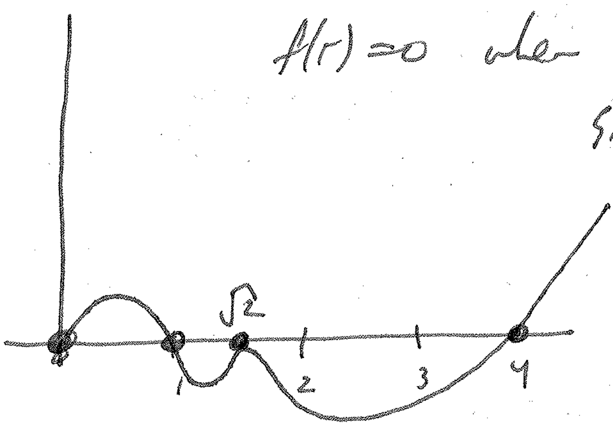
Question 4. [20 points] Do the following for the system of ODEs in polar coordinates given by

$$\dot{r} = (4-r)(r^2-2)^4(1-r)$$
$$\dot{\theta} = 1.$$

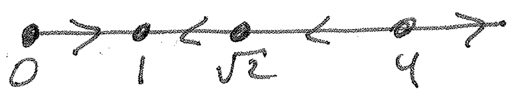
(a) Find and classify all limit cycles and equilibria.

(b) Sketch a rough graph of $x(t) = r(t)\cos\theta(t)$ as a function of t when $r(0) = 3$, $\theta(0) = 0$. (Note that you do not need exact values for the amplitudes. Just give a rough idea of how the solutions begins and how it eventually evolves.

Here, $f(r) = (4-r)(r^2-2)^4(1-r)$ has graph



$f(r)=0$ when $r = 0, 1, \sqrt{2}, 4$. Using the graph, we can construct a phase line

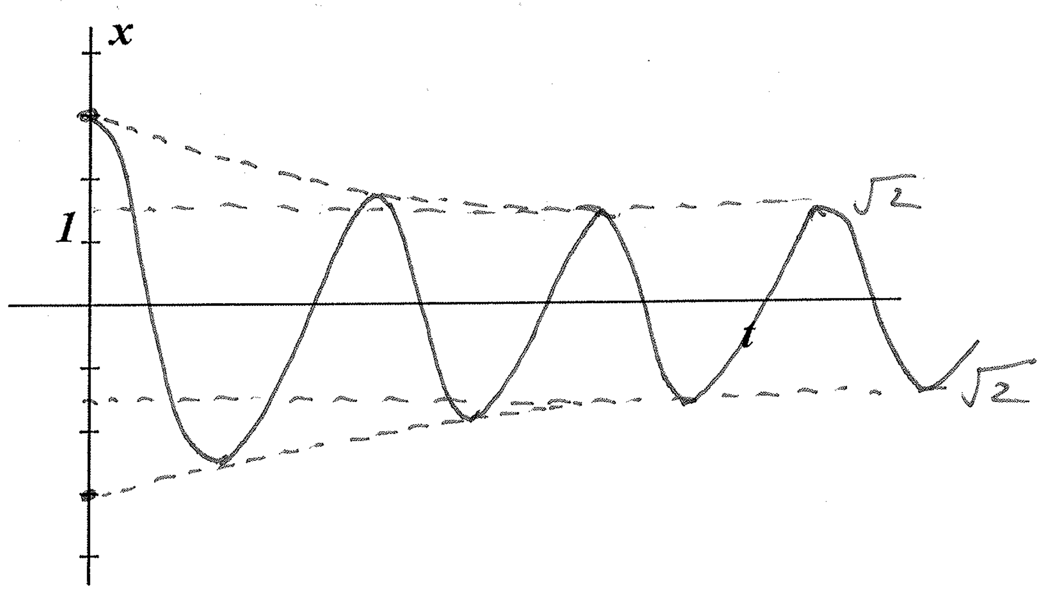


Here $r(t) \equiv 0$ is an unstable equilibrium,

$r(t) \equiv 1$ is a stable limit cycle,

$r(t) \equiv \sqrt{2}$ is semi-stable limit cycle

$r(t) \equiv 4$ is an unstable limit cycle.



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Question 5. [25 points] The system

$$\dot{x} = 1 - 5x + x^2 y$$
$$\dot{y} = 4x - x^2 y$$

is almost linear everywhere, and has a unique equilibrium. Do the following:

(a) Linearize the system at the unique critical point.

(b) Classify the type and stability of the origin of the associated linear system.

(c) Use the information form **Part (b)** to, as best as you can, classify the type and stability of the non-linear equilibrium (you must justify why or why not the non-linear equilibrium can be classified this way).

(a) Critical pt location: (a) $1-5x+x^2y=0$, $4x-x^2y=0$

$(4-xy)x=0$

Using bottom eqn, suppose $x=0$. Then top eqn reduces to $1=0$. Impossible.

Hence $x\neq 0$. But then bottom eqn implies $4-xy=0$.

or $4=xy$.

Sub this into top eqn to get

$$0=1-5x+x^2y=1-5x+x(4)=1-x$$

$\Rightarrow \boxed{x=1}$.

Then since $4=xy$, $\Rightarrow \boxed{y=4}$.

Critical @ $(1,4)$

Calculate $A = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(1,4)} = \begin{bmatrix} -5+2xy & x^2 \\ 4-2xy & -x^2 \end{bmatrix}\Bigg|_{(1,4)} = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$.

Hence assoc. linear system is

$$\vec{x}' = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} \vec{x}$$

(b) Eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ satisfy.

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

Here $r=1$ is a repeated eigenvalue,

The eigenvector(s) satisfy

$$\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \left.\begin{matrix} 3v_1 + v_2 = v_1 \\ -4v_1 - v_2 = v_2 \end{matrix}\right\} 2v_1 = -v_2$$

and there is only 1 linearly indep eigenvector.

$\Longrightarrow$ origin is an <u>improper source.</u>

(c) By classification thm, we cannot classify the type; it may be a ~~node or~~ proper or improper node.

But we do know that it is <u>unstable.</u>

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Possibly helpful formulae

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $W(y_1, y_2) = y_1 y_2' - y_1' y_2$
- $y_1 v'' + \left(2y_1' + p y_1\right) v' = 0$
- $A\vec{v} = r\vec{v}, \quad (A - rI)\,\vec{w} = \vec{v}$
- $\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$
- $y(t) = c_1 \vec{v} e^{rt} + c_2 \left(\vec{w} e^{rt} + \vec{v} t e^{rt}\right)$
- $Y(t) = u_1 y_1 + u_2 y_2$
- $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$
- $y(t) = c_1 e^{\lambda t}\left(\vec{a}\cos\mu t - \vec{b}\sin\mu t\right) + c_2 e^{\lambda t}\left(\vec{a}\sin\mu t + \vec{b}\cos\mu t\right)$
- $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Coefficients Table:	
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$	$t^s\left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right)$
$P_n(t) e^{\alpha t}$	$t^s\left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s\left[\left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right) e^{\alpha t}\sin\beta t + \left(B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0\right) e^{\alpha t}\cos\beta t\right]$

Laplace Transforms:	
$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
t^n, $(n = 0, 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin at$	$\frac{a}{s^2 + a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2 + a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2 - a^2}$, $s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2 - a^2}$, $s > \lvert a\rvert$
$t \sin at$	$\frac{2as}{(s^2 + a^2})^2$, $s > 0$
$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2})^2$, $s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$
$e^{at} f(t)$	$F(s-a)$, $s > a$