Name: ______________________ **Section Number:** __________

110.302 Differential Equations
FALL 2012
MIDTERM EXAMINATION C
December 5, 2012

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		20
3		20
4		20
5		25
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________ **Date:** __________

PLEASE SHOW ALL WORK, EXPLAIN YOUR REASONS, AND STATE ALL THEOREMS YOU APPEAL TO

Question 1. [15 points] Suppose $y(t)$ is the solution to the IVP $y' = 2(t - y^2)$, $y(0) = 1$. Using the Euler Method with a step size of $\frac{1}{2}$, estimate $y(2)$.

Euler Method: $t_{n+1} = t_n + h$
$y_{n+1} = y_n + h f(t_n, y_n)$

Here $t_0 = 0$, $y(t_0) = y(0) = 1$.

We calculate:

$\boxed{\text{Solution: For } y(t) \text{ a solution to the IVP, } y(2) \approx \frac{19}{16}}$

$t_1 = .5 = \frac{1}{2}$

$y_1 = y(.\frac{1}{2}) = y_0 + h f(t_0, y_0)$
$= 1 + \frac{1}{2}(2(0 - (1)^2)) = 1 + \frac{1}{2}(-2) = 0$

$t_2 = 1$

$y_2 = y(1) = y_1 + h f(t_1, y_1) = 0 + \frac{1}{2}(2(\frac{1}{2} - (0)^2))$
$= \frac{1}{2}$

$t_3 = \frac{3}{2}$

$y_3 = y(\frac{3}{2}) = y_2 + h f(t_2, y_2) = \frac{1}{2} + \frac{1}{2}(2(1 - (\frac{1}{2})^2))$
$= \frac{1}{2} + \frac{1}{2}(2(\frac{3}{4})) = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

$t_4 = 2$, $y_4 = y(2) = y_3 + h f(t_3, y_3) = \frac{5}{4} + \frac{1}{2}(2(\frac{3}{2} - (\frac{5}{4})^2))$
$= \frac{5}{4} + (\frac{3}{2} - \frac{25}{16}) = \frac{5}{4} + (\frac{-1}{16}) = \frac{19}{16}.$

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Question 2. [20 points] Do the following:

(a) Solve the Initial Value Problem

$$(*) \qquad 4y'' - 4y' - 8y = 8e^{-t}, \quad y(0) = 1, \; y'(0) = -\frac{2}{3}.$$

(b) Using the Method of Undetermined Coefficients, what would your guess of a particular solutions to the non-homogeneous ODE be if the forcing function (the right-hand-side) of the equation was $8e^{-t}(\sin 2t) + 8t^2$?

(a) First, we solve for the gen. soln of the homogeneous eqn $4y''-4y'-8y=0$. Characteristic eqn is

$$r^2 - r - 2 = 0 = (r-2)(r+1)$$

Since roots $r_1 = 2$, $r_2 = -1$ are real, distinct.

Homog. gen soln is $y(t) = c_1 e^{2t} - c_2 e^{-t}$.

Second: To find particular soln, use Method of UC:

Guess $Y(t) = Ate^{-t}$ (need t since e^{-t} is a homogeneous soln).

Here $Y'(t) = Ae^{-t} - Ate^{-t}$

$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

Sub into $(*)$ to get

$$4(-2Ae^{-t} + Ate^{-t}) - 4(Ae^{-t} - Ate^{-t}) - 8Ate^{-t} = 8e^{-t}$$

e^{-t} eqn: $-8A - 4A = 8 \Rightarrow A = \frac{8}{-12} = -\frac{2}{3}$.

te^{-t} eqn: $4A + 4A - 8A = 0$

General soln is: $y(t) = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3}te^{-t}$

Initial data:

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - c_2 - \cancel{2}/\cancel{3} = -\cancel{4}/3 \implies 2c_1 = c_2$$

Put these together to get

$$c_1 + c_2 = c_1 + 2c_1 = 1 \implies c_1 = \frac{1}{3}$$

$$\implies c_2 = \frac{2}{3}$$

Hence ~~general soln is~~ soln to the IVP is

$$\boxed{y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} - \frac{2}{3}te^{-t}}$$

(B) 16 Forcing function is $g(t) = 8e^{-t}(\sin 2t) + 8t^2$

A guess for the undetermined coefficient method is

$$\boxed{Y(t) = Ae^{-t}\sin 2t + Be^{-t}\cos 2t + Ct^2 + Dt + E}$$

Question 3. [20 points] For the second-order, linear, ODE

$$3p'' + 6p' + 15p = 0, \qquad (*)$$

do the following:

(a) Rewrite the equation as a system of two first-order ODEs.

(b) Find the general vector solution to the associated linear homogeneous system.

(c) Carefully draw the phase portrait for this system.

(a) Let $x = p$, $y = p'$. Then $(*)$ can be written as the system:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\tfrac{6}{3}p' - \tfrac{15}{3}p \end{aligned}\right\} \Rightarrow \left\{ \boxed{\begin{aligned} \dot{x} &= y \\ \dot{y} &= -5x - 2y. \end{aligned}} \right.$$

(b) Here system has matrix

$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ with characteristic eqn $r^2 + 2r + 5 = 0$

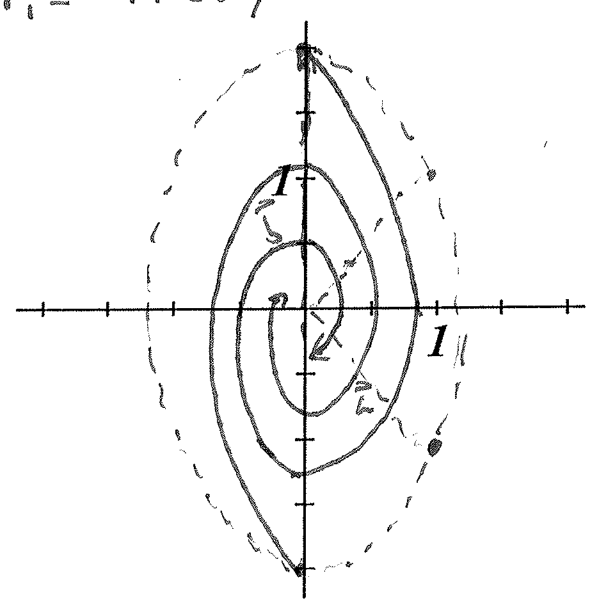
with roots $r = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

The complex eigenvalues have eigenvectors $\begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (-1 \pm 2i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

which solve $v_2 = (-1 \pm 2i)v_1$

Here $\vec{v}_1 = \begin{bmatrix} 1 \\ -1+2i \end{bmatrix}$, $r_1 = -1+2i$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1-2i \end{bmatrix}$, $r_2 = -1-2i$.

Thus
$\vec{v} = \vec{a} + i\vec{b}$
$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



(c) General elliptic basis is shown. Direction from $\vec{a}$ to $-\vec{b}$ shown.

Origin is a spiral sink.

general vector soln is $\vec{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin 2t\right)$
$+ c_2 e^{-t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos 2t\right).$

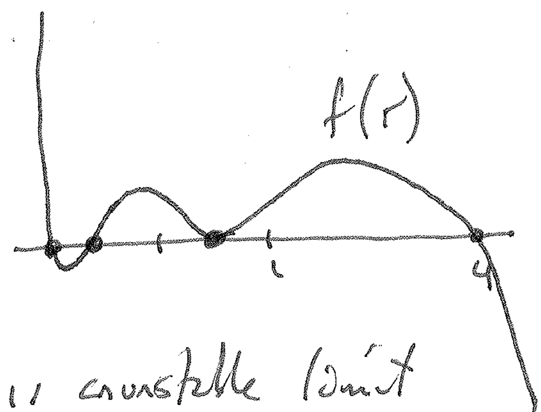
Question 4. [20 points] Do the following for the system of ODEs in polar coordinates given by

$$\dot{r} = (4-r)(r-.25)(r^2-2)^4$$
$$\dot{\theta} = -1.$$

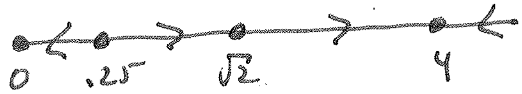
(a) Find and classify all limit cycles and equilibria.

(b) Sketch a rough graph of $x(t) = r(t)\cos\theta(t)$ as a function of t when $r(0) = .5$, $\theta(0) = 0$. (Note that you do not need exact values for the amplitudes. Just give a rough idea of how the solutions begins and how it eventually evolves.

(a) Possible equilibria and/or limit cycles correspond to places where $f(r) = (4-r)(r-.25)(r^2-2)^4 = 0$ or when $r = 0, .25, \sqrt{2}, 4.$

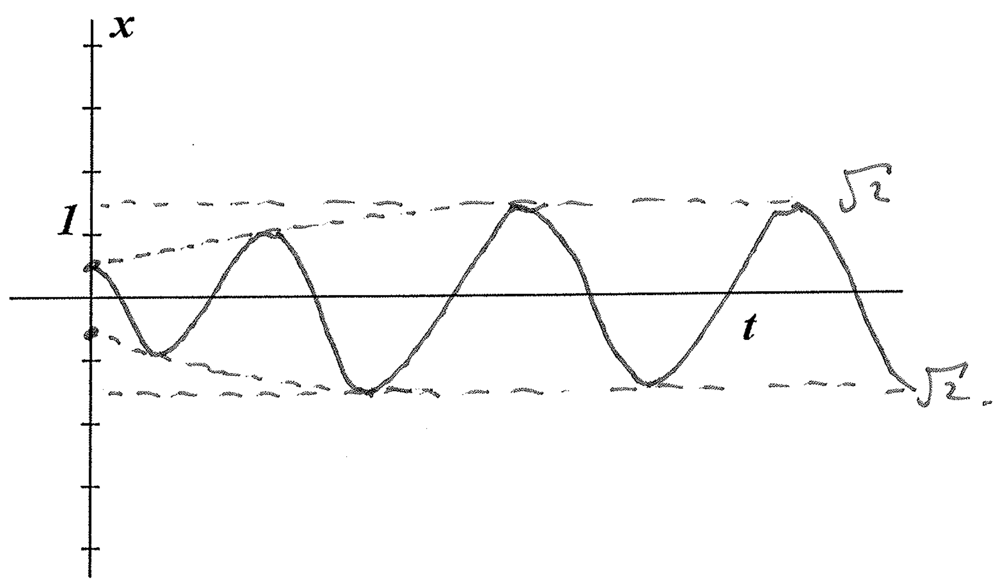


Phase line is



Here origin is a sink, $r(t) \equiv \frac{1}{4}$ is an unstable limit cycle, $r(t) \equiv \sqrt{2}$ is semi-stable limit cycle, and $r(t) \equiv 4$ is an ~~un~~stable limit cycle.

(b)



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Question 5. [25 points] The system

$$\dot{x} = 4x - x^2y - 1$$
$$\dot{y} = x^2y - 3x$$

is almost linear everywhere, and has a unique equilibrium. Do the following:

(a) Linearize the system at the unique critical point.

(b) Classify the type and stability of the origin of the associated linear system.

(c) Use the information form **Part (b)** to, as best as you can, classify the type and stability of the non-linear equilibrium (you must justify why or why not the non-linear equilibrium can be classified this way).

(a) Critical values satisfy $4x - x^2y - 1 = 0$

$x^2y - 3x = x(xy-3) = 0.$

If $x=0$, then top eqn is $-1=0$. Cannot happen.

If $y=0$, then bottom eqn means $x=0$. Cannot happen

Hence $x \neq 0$. But then by bottom eqn $xy-3=0$

So $\underline{xy=3}$. Thus top eqn is $4x - x(3) - 1 = 0$

or $x - 1 = 0$

$\Rightarrow \boxed{x=1}$. Then $(1)^2y - 3(1) = 0 \Rightarrow \boxed{y=3}$

Only critical value is $(1,3)$.

Calculate $A = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Big|_{(1,3)} = \begin{bmatrix} 4-2xy & -x^2 \\ 2xy-3 & x^2 \end{bmatrix}\Big|_{(1,3)} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$

Linear system is

$$\bar{x}' = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} \bar{x}.$$

(B) The char eqn of A is $r^2+r+1=0$.

with roots $r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Hence origin of linear system is a spiral sink.

(C) By classification Thm, the nonlinear equilibrium is also a spiral sink at $(1,3)$.

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Possibly helpful formulae

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $W(y_1, y_2) = y_1 y_2' - y_1' y_2$
- $y_1 v'' + \left(2y_1' + p y_1\right) v' = 0$
- $A\vec{v} = r\vec{v}, \quad (A - rI)\,\vec{w} = \vec{v}$
- $\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$
- $y(t) = c_1 \vec{v} e^{rt} + c_2 \left(\vec{w} e^{rt} + \vec{v} t e^{rt}\right)$
- $Y(t) = u_1 y_1 + u_2 y_2$
- $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$
- $y(t) = c_1 e^{\lambda t} \left(\vec{a} \cos \mu t - \vec{b} \sin \mu t\right) + c_2 e^{\lambda t} \left(\vec{a} \sin \mu t + \vec{b} \cos \mu t\right)$

pt $y_{n+1} = y_n + h f(t_n, y_n)$

- $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Coefficients Table:	
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$	$t^s \left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right)$
$P_n(t) e^{\alpha t}$	$t^s \left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s \left[\left(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\right) e^{\alpha t} \sin \beta t + \left(B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0\right) e^{\alpha t} \cos \beta t\right]$

Laplace Transforms:	
$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
t^n, $(n = 0, 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin at$	$\frac{a}{s^2 + a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2 + a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2 - a^2}$, $s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2 - a^2}$, $s > \lvert a \rvert$
$t \sin at$	$\frac{2as}{(s^2 + a^2}\,)^2$, $s > 0$
$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2}\,)^2$, $s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$
$e^{at} f(t)$	$F(s-a)$, $s > a$