Name: ______________________________ **Section Number:**__________

110.302 Differential Equations
FALL 2012
MIDTERM EXAMINATION B SOLUTIONS
October 12, 2012

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		30
2		15
3		15
4		20
5		20
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**__________

Question 1. [30 points] Solve the following first-order differential equations:

(a) $\dot{x} + (xt)^2 = x^2 e^{3t}$.

Strategy: This first-order ODE is NOT linear, but it is separable. We place it in a form to use the separation of variables techniques to integrate and solve.

Solution: If we solve for the derivative, we get

$$\frac{dx}{dt} = x^2 e^{3t} - (xt)^2 = x^2 e^{3t} - x^2 t^2 = x^2(e^{3t} - t^2).$$

Separating the variables yields $\frac{1}{x^2}\frac{dx}{dt} = e^{3t} - t^2$, and integrating with respect to t gives us

$$\int \frac{1}{x^2}\frac{dx}{dt}\, dt = \int \frac{1}{x^2}\, dx = \int (e^{3t} - t^2)\, dt$$

$$-\frac{1}{x} = \frac{1}{3}e^{3t} - \frac{t^3}{3} + K = \frac{e^{3t} - t^3 + C}{3}.$$

Note here that the K is the constant of integration. To be precise, creating the single fraction requires a different constant in the numerator. Here $C = 3K$, but this is a small point, Now invert both sides of this equation (if two numbers are equal, then their reciprocals are also), and

$$\boxed{x(t) = \frac{-3}{e^{3t} - t^3 + C}.}$$

Now, did you catch the missed solution? Upon separating the variables, we divided by x^2. This assumes that x is not 0. But $x(t) \equiv 0$ IS a solution. It is extraneous due to the method of solution (Notice that you cannot find a value of C which gives this solution.) Hence it must be added in. Hence also

$$\boxed{x(t) \equiv 0.}$$

(b) $\left(x^2 + 1\right)\frac{dy}{dx} + xy - x = 0, \quad y(0) = -2.$

Strategy: This first-order is linear. We put it in standard form, and then use an integrating factor to solve.

Solution: Putting this ODE into standard form, we get $\frac{dy}{dx} + \left(\frac{x}{x^2+1}\right) y = \frac{x}{x^2+1}$. Note that here $p(x) = \frac{x}{1+x^2}$, so calculating the integration factor yields

$$e^{\int p(x)\, dx} = e^{\int \frac{x}{1+x^2}\, dx} = e^{\frac{1}{2}\int \frac{1}{u}\, du} = e^{\frac{\ln u}{2}} = e^{\ln \sqrt{u}} = \sqrt{u} = \sqrt{x^2+1},$$

upon using the substitution $u = x^2 + 1$. Multiplying through the entire ODE by the integrating factor, we get

$$\sqrt{x^2+1}\left[\frac{dy}{dx} + \left(\frac{x}{x^2+1}\right) y = \frac{x}{x^2+1}\right]$$

$$\sqrt{x^2+1}\frac{dy}{dx} + \left(\frac{x}{\sqrt{x^2+1}}\right) y = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left[\left(\sqrt{x^2+1}\right) y\right] = \frac{x}{\sqrt{x^2+1}}.$$

Now we integrate directly with respect to x and get

$$\left(\sqrt{x^2+1}\right) y = \sqrt{x^2+1} + C, \quad \Longrightarrow \quad y(t) = 1 + \frac{C}{\sqrt{x^2+1}}.$$

For the initial value, note that $y(0) = 1 + \frac{C}{\sqrt{(0)^2+1}} = 1 + C = -2, \quad \Longrightarrow \quad C = 3.$

Hence $\boxed{y(t) = 1 + \frac{3}{\sqrt{x^2+1}}.}$

Question 2. [15 points] Solve the Initial Value Problem

$$y^2 + x^2 - 2x + 3 + \left(2xy - y^2 + 10\right)\frac{dy}{dx} = 0, \quad \text{where} \quad y(0) = 3.$$

Strategy: We first show that this first-order ODE is exact. We will use the standard form for an exact ODE, $M(x, y) + N(x, y)y' = 0$, so integrate to find the function of two variables $\varphi(x, y)$ whose partial derivatives are M and N. The level sets of this function $\varphi(x, y)$ are integral curves of the ODE, and express y as an implicit function of x. We then use the initial data to find the proper level set.

Solution: In standard form, we find

$$M(x, y) = y^2 + x^2 - 2x + 3, \quad \text{and} \quad N(x, y) = 2xy - y^2 + 10.$$

This ODE is exact since

$$M_y = \frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} = N_x.$$

Hence solving this ODE means integrating M and N under the understanding that $M = \frac{\partial \varphi}{\partial x}$ and $N = \frac{\partial \varphi}{\partial y}$. Integrating the first expression, we get

$$\varphi(x, y) = \int \frac{\partial \varphi}{\partial x}\, dx = \int M\, dx = \int \left(y^2 + x^2 - 2x + 3\right)\, dx = xy^2 + \frac{x^3}{3} - x^2 + 3x + h(y),$$

where the unknown function $h(y)$ is our constant of integration (it is a constant with respect to x). We can now partially differentiate $\varphi(x, y)$ with respect to y and compare to $N(x, y)$:

$$\frac{\partial}{\partial y}\left[\varphi(x, y)\right] = \frac{\partial}{\partial y}\left[xy^2 + \frac{x^3}{3} - x^2 + 3x + h(y)\right] = 2xy + h'(y) = 2xy - y^2 + 10 = N(x, y).$$

Hence it must be the case that $h'(y) = -y^2 + 10$, so that $h(y) = -\frac{y^3}{3} + 10y$. Hence $\varphi(x, y) = xy^2 + \frac{x^3}{3} - x^2 + 3x - \frac{y^3}{3} + 10y$ and the level sets of $\varphi(x, y)$ comprise the integral curves of the ODE. Thus the general solution $y(x)$ is known only implicitly as

$$xy^2 + \frac{x^3}{3} - x^2 + 3x - \frac{y^3}{3} + 10y = C.$$

Using the initial data $x = 0$ and $y = 3$, we get

$$(0)(3)^2 + \frac{(0)^3}{3} - (0)^2 + 3(0) - \frac{(3)^3}{3} + 10(3) = C. = 21.$$

Hence the (implicit) particular solution to this exact ODE is

$$\boxed{xy^2 + \frac{x^3}{3} - x^2 + 3x - \frac{y^3}{3} + 10y = 21.}$$

Question 3. [15 points] For the second-order, linear, homogeneous Initial Value Problem:

$$3y'' + 18y' + 27y = 0, \quad y(0) = 2, \quad y'(0) = a,$$

do the following:

(a) Find the particular solution (Hint: Your answer will have an a in it).

Strategy: This second-order linear homogeneous ODE has constant coefficients. We use the roots of the corresponding characteristic equation to write out the general solution. Then we use the initial data to solve for the unknown constants.

Solution: The characteristic equation for this ODE is

$$3r^2 + 18r + 27 = 0 = r^2 + 6r + 9 = (r+3)^2.$$

The only root of this quadratic equation is $r = -3$. It has a multiplicity of 2. We can now write out the solutions directly using the formula:

$$y(t) = c_1 e^{rt} + c_2 te^{rt} = c_1 e^{-3t} + c_2 te^{-3t}.$$

Using the initial data, we get $y(0) = c_1 e^{-3(0)} + c_2(0)e^{-3(0)} = 2$, so that $c_1 = 2$. Then

$$\begin{aligned} y'(0) &= \frac{d}{dt}\bigg|_{t=0} \left[2e^{-3t} + c_2 te^{-3t}\right] \\ &= \left(-3(2)e^{-3t} + c_2 e^{-3t} - 3c_2 te^{-3t}\right)\bigg|_{t=0} \\ &= -3(2)e^{-3(0)} + c_2 e^{-3(0)} - 3c_2(0)e^{-3(0)} = -3(2) + c_2 = a, \end{aligned}$$

so that $c_2 = (a+6)$. Hence we put this together to get out fundamental set of solutions

$$\boxed{y(t) = 2e^{-3t} + (a+6)te^{-3t}.}$$

(b) Determine for which values of a this solution will tend to 0 as $t \to \infty$.

Strategy: We use standard calculus techniques to determine when this function $y(t)$ has $x = 0$ as a horizontal asymptote.

Solution: We determine the values for a for which we get $\lim\limits_{t\to\infty} y(t) = 0$. We get

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 2e^{-3t} + (a+6)te^{-3t}.$$

This limit can be broken up into the sum of the limits IF the two individual limits exist. We get

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} 2e^{-3t} + (a+6)te^{-3t} \\ &= \lim_{t\to\infty} \frac{2}{e^{3t}} + \lim_{t\to\infty} \frac{(a+6)t}{e^{3t}} \\ &= 2\lim_{t\to\infty} \frac{1}{e^{3t}} + (a+6)\lim_{t\to\infty} \frac{t}{e^{3t}}. \end{aligned}$$

The first of the two limits exists and is 0 since e^{-3t} certainly has a horizontal asymptote at $y = 0$. But the second term also has a limit (Use L'Hospital's Rule on the ratio noting that the ration displays the indeterminate form $\frac{\infty}{\infty}$). Hence

$$\lim_{t\to\infty} y(t) = 2\lim_{t\to\infty} \frac{1}{e^{3t}} + (a+6)\lim_{t\to\infty} \frac{t}{e^{3t}} = 2(0) + (a+6)(0) = 0$$

regardless of the vlaue of the constant $a \in \mathbb{R}$. Hence for all values of $a \in \mathbb{R}$, $\lim\limits_{t\to\infty} y(t) = 0$.

Question 4. [20 points] Given the differential equation $3t^2y''+9ty'-9y=0$, defined on the interval $t>0$, and a known solution $y(t)=t^{-3}$, find a fundamental set of solutions.

Strategy: We use the Reduction of Order method to solve this problem. This involves assuming a solution of the form $y(t)=u(t)y_1(t)$, and using the differential equation itself to create a new ODE for which $u(t)$ is a solution. We already have a form for this (on last page, where we just need to identify $p(t)$ to make the proper substitution.

Solution: Given that $y_1(t)=t^{-3}$ solves the ODE, we assume a new solution is of the form $y_2(t)=u(t)t^{-3}$. Substituting this and its derivatives into the ODE leads to a new differential equation of the form

$$y_1v''+\left(2y_1'+py_1\right)v'=0,$$

where $y_1(t)=t^{-3}$, $y_1\prime(t)=-3t^{-4}$, and $p(t)=\frac{3}{t}$. We get

$$\begin{aligned}
\left(\frac{1}{t^3}\right)v''+\left(2\left(\frac{-3}{t^4}\right)+\left(\frac{3}{t}\right)\left(\frac{1}{t^3}\right)\right)v'&=0\\
\left(\frac{1}{t^3}\right)v''+\left(\frac{-3}{t^4}\right)v'&=0\\
tv''-3v'&=0\\
v''-\frac{3}{t}v'&=0.
\end{aligned}$$

This is a first-order ODE in $v'(t)$. We solve it by using an integration factor: Identify the integration factor as

$$e^{\int\left(-\frac{3}{t}\right)\,dt}=e^{-3\ln t}=e^{\ln t^{-3}}=t^{-3}.$$

Thus

$$\begin{aligned}
\frac{1}{t^3}\left[v''-\left(\frac{3}{t}\right)v'=0\right]\\
\frac{1}{t^3}v''-\left(\frac{3}{t^4}\right)v'&=0\\
\frac{d}{dt}\left[\frac{1}{t^3}v'\right]&=0\\
\frac{1}{t^3}v'&=C\\
v'(t)&=Ct^3.
\end{aligned}$$

Hence $v(t)=t^4$. The constant will not matter, since we will be taking linear combinations of this new function with the original solution in a minute. Hence

$$y_2(t)=v(t)y_1(t)=t^4t^{-3}=t.$$

Does this satisfy the ODE? By substitution, we get $3t^2y_2''+9ty_2'-9y_2=3t^2(0)+9t(1)-9(t)=0$. Is this solution independent form the other one? Calculate the Wronskian to see.

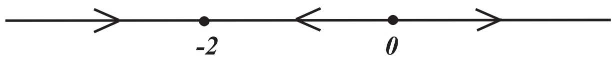
Hence our fundamental set of solutions is

$$\boxed{y(t)=c_1t^{-3}+c_2t.}$$

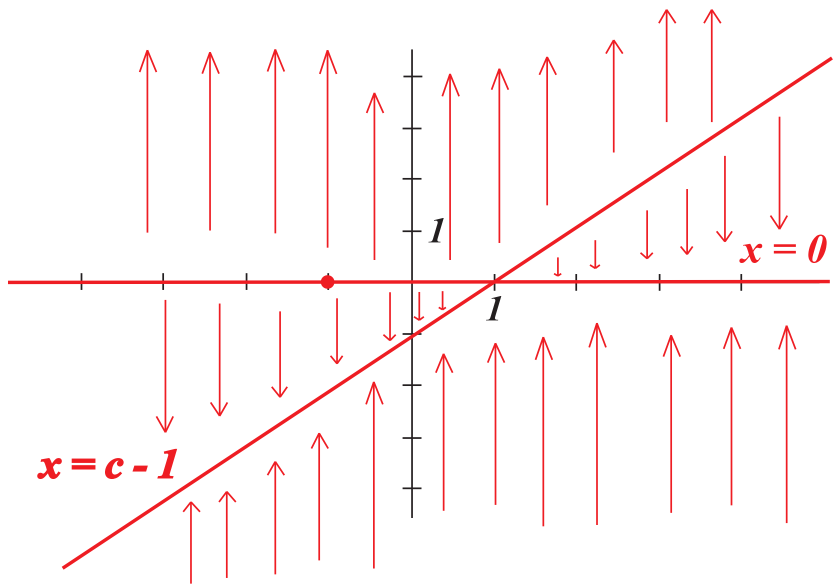
Question 5. [20 points] Let $\dfrac{dx}{dt} = x - x(c-x)$ be a first-order, autonomous differential equation, where c is a parameter. Do the following:

(a) Draw a phase line for the parameter value $c = -1$, and determine the stability of all of the equilibria.

Solution: Let $c = -1$, and find the places where the right-hand-side $f(x) = x - x(c-x) = 0$. Then determine the sign of $f(x)$ outside of these points, and draw the line. Here, for $c = -1$, we get $f(x) = x - x(-1-x) = 2x + x^2 = x(x+2)$. Here, $f(x) = 0$ when $x = 0$ or $x = -2$. When $x \in (-\infty, -2)$, $f(x) > 0$. When $x \in (-2, 0)$, $f(x) < 0$, and when $x \in (0, \infty)$, $f(x) > 0$. Hence the phase line is

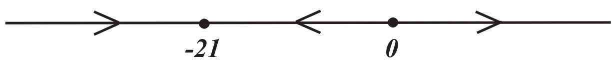


(b) Sketch on the graph below a bifurcation diagram for this ODE, indicating the stability of the fixed points and indicating ALL of the bifurcation values of c.



(c) Determine the long term behavior of the solution to this ODE passing through the point $x(0) = -2$, when $c = -20$ (that is, find $\lim_{t\to\infty} x(t)$ with this initial data.)

Solution: For $c = -20$, The ODE looks like $\dot{x} = x - x(-20-x) = 21x + x^2 = x(21+x)$. For this value of c, the phase line looks like



From this, it is easy to see that if the initial point is $x(0) = -2$, then the long term behavior tends to the equilibrium solution $x(t) = -21$. Hence

$$\lim_{t\to\infty} x(t) = -21$$

for this starting value.

Possibly helpful formulae

- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $W(y_1, y_2) = y_1 y_2' - y_1' y_2$
- $y_1 v'' + (2y_1' + p y_1)\, v' = 0$