HOMEWORK PROBLEM SET 7: DUE OCTOBER 22, 2018

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. Solve the following:

(a) $y'' + 2y' + y = \dfrac{e^{-x}}{x}$.

(b) $\ddot{r} + r = \csc t$.

Question 2. One can use the Reduction of Order ideas for a nonhomogeneous ODE

$$y'' + p(t)y' + q(t)y = g(t),$$

when one solution y_1 of the corresponding homogeneous version of the ODE is known; again assume $y_2(t) = v(t)y_1(t)$ also solves the nonhomogeneous ODE and substitute it and its derivatives back into the ODE, creating a first-order ODE in v', as before. Do this for the ODE

$$ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t.$$

Question 3. Consider the ODE

$$ay'' + by' + cy = g(t),$$

where a, b, c are positive constants. Do the following:

(a) If $Y_1(t)$ and $Y_2(t)$ are both solutions, show that $Y_1(t) - Y_2(t) \longrightarrow 0$ as $t \to \infty$. Is this still true if $b = 0$?

(b) If $g(t) = d$, a constant, show that every solution of the ODE approaches the constant $\frac{d}{c}$ as $t \to \infty$. What happens when $c = 0$? What happens if $b = c = 0$?

Question 4. A car supported by a MacPherson strut (shock absorber system) travels on a bumpy road at a constant velocity v. The equation modeling the motion of the car is

$$80\ddot{x} + 10000x = 2500\cos\left(\frac{\pi v t}{6}\right).$$

where x represents the vertical position of the cars axle relative to its equilibrium position, and the basic units of measurement are feet and feet per second (this is actually just an example of a forced, un-damped harmonic oscillator, if that is any help). The constant numbers above are related to the characteristics of the car and the strut. Note that the coefficient of time t (inside the cosine) in the forcing term on the right hand side is a frequency, which in this case is directly proportional to the velocity v of the car.

(a) Find the general solution to this nonhomogeneous ODE. Note that your answer will have a term in it which is a function of v.

(b) Determine the value of v for which the solution is undefined (you should present your final answer in miles per hour, as opposed to feet per second).

(c) For a set of initial values $x(0) = \dot{x}(0) = 0$, graph the solutions for a few values of v near your answer in part **(b)** and not so near. Discuss the differences in these graphs and the importance of the special value of v in part **(b)**. (Hint: This special value of v induces what is called resonance in the car).

Question 5. Solve the following:

(a) $y''' - 3y'' + 3y' - y = 0$.

(b) $\dfrac{d^4y}{dx^4} = 4\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2}$.

(c) $\dddot{x} - \ddot{x} + \dot{x} - x = 0, \quad x(0) = 2, \quad \dot{x}(0) = -1, \quad \ddot{x}(0) = -2$.

(d) $y''' - y'' - y' + y = 2e^{-t} + 3$.

Question 6. Determine the interval on which the solution to the IVP

$$(x-1)y^{(12)} + (x+1)y^{(4)} = (\tan x)y, \quad y(0) = 1, \quad y'(0) = \cdots = y^{(4)}(0) = 0,$$
$$y^{(5)}(0) = -6, \quad y^{(6)}(0) = \cdots = y^{(11)}(0) = \pi$$

is sure to exist and be unique.

Question 7. Verify that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the solution guess $y = v(t)y_1(t)$ leads to the following second order linear homogeneous ODE in v':

$$y_1 v''' + \left(3y_1' + p_1 y_1\right) v'' + \left(3y_1'' + 2p_1 y_1' + p_2 y_1\right) v' = 0.$$