

EXAMPLE: LINEAR DIFFERENTIAL EQUATIONS

110.302 DIFFERENTIAL EQUATIONS
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Problem. Solve the Ordinary Differential Equation $t(y' - y) = (1 + t^2)e^t$.

Strategy. Solving the ODE means finding the general solution (the 1-parameter family of solutions). We place this linear, first order ODE into its standard form and compute the integrating factor. Then we multiply the ODE by its integrating factor and then integrate to find the general solution.

Solution. This ODE is almost in the standard form $y' + p(t)y = q(t)$. To get there, simply divide the entire equation by the coefficient of y' which in this case is t . We get

$$\begin{aligned}\frac{1}{t} [t(y' - y) &= (1 + t^2)e^t] \\ y' - y &= \frac{1 + t^2}{t}e^t.\end{aligned}$$

Here then, $p(t) = -1$, and the integrating factor is

$$e^{\int p(t) dt} = e^{\int -1 dt} = e^{-t}.$$

Multiply this through the ODE to get

$$\begin{aligned}e^{-t} \left[y' - y \right] &= \frac{1 + t^2}{t}e^t \\ e^{-t}y' - e^{-t}y &= \frac{1 + t^2}{t}e^t.\end{aligned}$$

Note. While this new expression certainly looks harder to play with, it winds up making the problem easier to solve.

Continuing on, we get

$$\begin{aligned}
 e^{-t}y' - e^{-t}y &= \frac{1+t^2}{t} \\
 \frac{d}{dx} [e^{-t}y] &= \frac{1+t^2}{t} \\
 \int \frac{d}{dt} [e^{-t}y] dt &= \int \frac{1+t^2}{t} dt = \int \left(\frac{1}{t} + t \right) dt \\
 e^{-t}y &= \ln |t| + \frac{t^2}{2} + C \\
 y(t) &= e^t \left(\ln |t| + \frac{t^2}{2} + C \right).
 \end{aligned}$$

This is the general solution.

Is it correct? We check: Given $y(t)$ as above, we calculate, using the Product Rule,

$$y'(t) = e^t \left(\ln |t| + \frac{t^2}{2} + C \right) + e^t \left(\frac{1}{t} + t \right).$$

Substitute both $y(t)$ and $y'(t)$ into the original linear ODE, and we get

$$\begin{aligned}
 t(y' - y) &= (1+t^2)e^t \\
 t \left(\left(e^t \left(\ln |t| + \frac{t^2}{2} + C \right) + e^t \left(\frac{1}{t} + t \right) \right) - e^t \left(\ln |t| + \frac{t^2}{2} + C \right) \right) &= (1+t^2)e^t \\
 te^t \ln |t| + e^t \frac{t^3}{2} + Cte^t + e^t + t^2e^t - te^t \ln |t| - e^t \frac{t^3}{2} - Cte^t &= e^t + t^2e^t = (1+t^2)e^t.
 \end{aligned}$$

It all works.