EXAMPLE: EXACT DIFFERENTIAL EQUATIONS

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Problem. Solve the Initial Value Problem $2x + y^2 + 2xy\dfrac{dy}{dx} = 0$, $y(1) = 1$.

Strategy. Solving this ODE with an initial point means finding the particular solution to the ODE that passes through the point $(1,1)$ in the ty-plane. Here we show that the ODE is exact, and use standard calculus integration and differentiation to find a function of both x and y whose level sets are the implicit general solutions to the ODE. We then use the initial data to find the particular solution.

Solution. This ODE is exact. Indeed, we identify $M(x,y) = 2x + y^2$ as the collection of all terms not attached to the y' factor, and $N(x,y) = 2xy$ as the coefficient of y'. Then the exactness condition is

$$\frac{\partial M}{\partial y} = M_y = N_x = \frac{\partial N}{\partial x}$$
$$2y = 2y.$$

Thus we know by theorem that there exists a function $\varphi(x,y)$, where (1) $\frac{\partial \varphi}{\partial x} = M$, (2) $\frac{\partial \varphi}{\partial y} = N$, and (3) $\varphi(x,y) = C$ is the general solution to the exact ODE. We can recover this function $\varphi(x,y)$ by integrating the partial derivatives. Here, we fist integrate M to get some specific information on φ. Here

$$\int M\,dx = \int \left(\frac{\partial \varphi}{\partial x}\right) dx = \int \left(2x + y^2\right) = x^2 + xy^2 + h(y) = \varphi(x,y).$$

Thus we have a good idea of what φ looks like, up to the unknown function $h(y)$. Notice here, though, that the constant of integration may not be a constant. This is because you are finding the antiderivative of a partial derivative. With respect to x, the antiderivatives of M will vary by a constant in the x-variable. Thus ANY function of y alone would serve as a constant under the partial derivative with respect to x. We need to account for this in general. Hence the term consisting of the unknown function $h(y)$ at the end.

To continue, we now can use the other partial derivative to work out the rest of φ. Indeed, we use what we know to calculate

$$\frac{\partial}{\partial y}\varphi(x,y) = \frac{\partial}{\partial y}\left[x^2 + xy^2 + h(y)\right] = 2xy + h'(y).$$

Here, $h(y)$ is a function ONLY of y, so the partial derivative IS the total derivative. This last expression for the partial of φ with respect to y also IS N, so that

$$\frac{\partial}{\partial y}\varphi(x,y) = 2xy + h'(y) = 2xy = N.$$

Hence $h'(y) = 0$, and we can conclude that $h(y)$ is a constant. Thus $\varphi(x,y) = x^2 + xy^2 + \text{ constant } = C$, or realizing that the two constants are really one constant since both are *a priori* unknown,

$$\varphi(x,y) = x^2 + xy^2 = C$$

is the general solution to the ODE $2x + y^2 + 2xyy' = 0$.

To solve the IVP, set $x = 1$ and $y = 1$, to get

$$\varphi(1,1) = (1)^2 + (1)(1)^2 = C, \quad \Longrightarrow \quad C = 2,$$

and our particular solution to the IVP is $x^2 + xy^2 = 2$, at least implicitly.

We should take this one step further and understand the domain for the solution. Solving for y, we get the integral curve defined by the pieces

$$y = \pm\sqrt{\frac{2-x^2}{x}}.$$

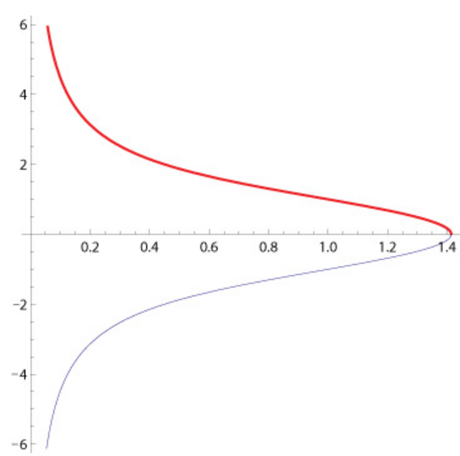
Limiting ourselves to the piece containing the point $(1,1)$, we get $y = \sqrt{\frac{2-x^2}{x}}$. The domain of this function is only $(0, \sqrt{2}]$. However, also keep in mind that the derivative of this function must also make sense for the ODE to even make sense. Hence we calculate the derivative and get

$$y'(x) = \frac{1}{2\sqrt{\frac{2-x^2}{x}}}\left(2\ln(x) - \frac{x^2}{2}\right).$$

The domain for this function is $(0, \sqrt{2})$. Hence the proper solution to this equation that fits the ODE is really only $(0, \sqrt{2})$. Hence the particular solution to this IVP is

$$y(x) = \sqrt{\frac{2-x^2}{x}}, \qquad \text{for } x \in (0, \sqrt{2}).$$

The graph of $y(x)$ is in red.



Is it correct? Check: For $y(x) = \sqrt{\frac{2-x^2}{x}}$, we have

$$y'(x) = \frac{1}{2}\left(\frac{2-x^2}{x}\right)^{-\frac{1}{2}} \cdot \left(-\frac{2}{x^2} - 1\right) = \frac{\frac{-2-x^2}{2x^2}}{\sqrt{\frac{2-x^2}{x}}}.$$

Thus the ODE is

$$\begin{aligned}
2x + y^2 + 2xyy' &= 0 \\
2x + \left(\sqrt{\frac{2-x^2}{x}}\right)^2 + 2x\left(\sqrt{\frac{2-x^2}{x}}\right)\left(\frac{\frac{-2-x^2}{2x^2}}{\sqrt{\frac{2-x^2}{x}}}\right) &= 0 \\
2x + \frac{2-x^2}{x} + 2x\left(\frac{-2-x^2}{2x^2}\right) &= 0 \\
2x + \frac{2}{x} - x - \frac{4}{2x} - x &= 0.
\end{aligned}$$

It all works.