CHALLENGE PROBLEM SET: WEEK 12

110.302 DIFFERENTIAL EQUATIONS
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Question 1. The undamped pendulum is modeled on the system $\frac{dx}{dt} = y$, and $\frac{dy}{dt} = -\omega^2 \sin x$. Do the following:

- **(a)** Find all critical points and show that the system is locally linear at each critical point.
- **(b)** Show that the linearized equilibrium at the origin is a center. Can one justify declaring that the nonlinear equilibrium is also a center at the origin? Which other equilibria have the same conclusion?
- **(c)** Show that the linearized equilibrium at the point $x = \pi$ and $y = 0$ is a saddle. Can one justify declaring that the corresponding nonlinear equilibrium is also a saddle? Which other equilibria have the same conclusion? What is the physical interpretation of these critical points? In particular, saddles are unstable. But what is the pendulum interpretation of the trajectories that tend toward the saddle as t goes to infinity?
- **(d)** For a particular non-zero value ω, draw some solutions near the origin? Can you draw any conclusions about the type of the nonlinear equilibrium at the origin based on your drawings?

Question 2. For the system $\dot{x} = y - x$ and $\dot{y} = \epsilon x - y$, where ϵ is a parameter, classify the type and stability of the origin for different values of ϵ, and use this to verify that an improper node is not structurally stable in terms of type, but it is in terms of stability; that the type of equilibrium at the origin does not persist under perturbations, even as the stability does.

Question 3. For each system, (1) find all critical points, (2) linearize the system at each critical point and, as best as you can, classify each equilibrium according to type and stability, using the linearized system. These are the same systems you constructed phase portraits for last week. Now overlay the linearized phase portraits over the top of the ones you drew last week.

- **(a)** $\frac{dx}{dt} = -x + 2xy, \quad \frac{dy}{dt} = y - x^2 - y^2.$
- **(b)** $\frac{dx}{dt} = (2-x)(y-x), \quad \frac{dy}{dt} = y(2 - x - x^2).$

Question 4. For the competing species model $\dot{x} = x\left(\frac{3}{2} - \frac{1}{2}x - y\right)$ and $\dot{y} = y\left(\frac{3}{4} - y - \frac{1}{8}x\right)$, draw a phase portrait and discuss the limiting behavior of the species populations $x(t)$ and $y(t)$ as $t \to \infty$ for various initial population sizes. Now do the same for the predator-prey model $\dot{x} = \frac{x}{2}(3 - y)$ and $\dot{y} = \frac{y}{2}(2x - 1)$.

Question 5. For the competing species model $\dot{x} = x\,(1 - x - y)$ and $\dot{y} = y\left(\alpha - y - \frac{x}{2}\right)$, do the following:

(a) Find all critical points (some of these may be functions of α).

(b) Determine any bifurcation points for α.

(c) Calculate the linearization matrix for each critical point.

(d) Determine the type and stability of each critical point via the linearization matrix and watch what happens at the bifurcation points of α.

(e) Draw phase portraits on either side of a bifurcation point for α to illustrate the changes in the phase portraits.

Question 6. For each ODE system, find and classify all equilibria and cycles and draw a phase portrait:

(a) $\dot{r} = \sqrt{1+r^2}(4r - 3 - r^2)(r - 2), \quad \dot{\theta} = 1.$

(b) $\dot{r} = r|1 - r|(r - 3), \quad \dot{\theta} = \pi.$