

CHALLENGE PROBLEM SET: WEEK 11

110.302 DIFFERENTIAL EQUATIONS
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Question 1. For the following systems, find a general solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$

(b) $\mathbf{x}' = \begin{bmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{bmatrix} \mathbf{x}.$

Question 2. Solve the IVP

$$\mathbf{x}' = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

and draw the solution in the phase plane. Also graph $x_1(t)$ as a function of t .

Question 3. For the following non-autonomous, first-order, homogeneous linear ODE systems, note that instead of assuming the exponential functions solve the system, note that solutions of the form $\mathbf{x} = \mathbf{v}t^r$ solve these systems, where the pair $r, \mathbf{v}$ are again eigenvalue/eigenvector pairs of the constant matrix of coefficients A . Use this to solve the system

$$t\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$$

Question 4. For the following systems, determine the critical point $\mathbf{x}(t) = \mathbf{x}^0$, and then classify its type and stability by making a change of variables transformation to the system $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$. Note that in the new variables $\mathbf{u}$, the origin is not the equilibrium. Then sketch a phase portrait for the original system.

(a) $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$

(b) $\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$

Question 5. For each system, (1) find all critical points, (2) use a computer or hand draw a direction field and sketch a few trajectories near the critical points, and (3) determine as best as one can the type and stability of each critical point.

(a) $\frac{dx}{dt} = -x + 2xy, \quad \frac{dy}{dt} = y - x^2 - y^2.$

(b) $\frac{dx}{dt} = (2 - x)(y - x), \quad \frac{dy}{dt} = y(2 - x - x^2).$

Question 6. For each system, (1) find a function $H(x, y) = c$ satisfied by the trajectories, and (2) plot several level curves of H , indicating the direction of travel for increasing t .

(a) $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = 2x + y.$

(b) $\frac{dx}{dt} = -x + y, \quad \frac{dy}{dt} = -x - y.$