

## CHALLENGE PROBLEM SET: WEEK 10

110.302 DIFFERENTIAL EQUATIONS  
PROFESSOR RICHARD BROWN

**Question 1.** For the following systems, find a general (real) solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a)  $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}.$

(b)  $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}.$

**Question 2.** Solve the IVP

$$\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and, in detail, describe the behavior of the solution, both near  $t = 0$  as well as when  $t$  goes to infinity and minus infinity. For instance, describe how close it gets to the origin and any possible asymptotes that it may have. If it has asymptotes, describe the nature of these asymptotes.

**Question 3.** For the ODE system

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix} \mathbf{x},$$

do the following:

- (a) Determine the eigenvalues as functions of  $\alpha$ .
- (b) Find the critical values of  $\alpha$ , defined as values of  $\alpha$  where the qualitative nature of the phase portrait for the system changes.
- (c) Draw the phase portrait for values of  $\alpha$  slightly larger and slightly smaller than each critical value of  $\alpha$ .

**Question 4.** For the following non-autonomous, first-order, homogeneous linear ODE systems, note that instead of assuming the exponential functions solve the system, note that solutions of the form  $\mathbf{x} = \mathbf{v}t^r$  solve these systems, where the pair  $r, \mathbf{v}$  are again eigenvalue/eigenvector pairs of the constant matrix of coefficients  $A$ . Do the following:

- (a) Show that this assertion is true: That if you assume (guess?) that a solutions exists of the form  $\mathbf{x} = \mathbf{v}t^r$ , then it will be true when  $r$  and  $\mathbf{v}$  satisfy the equation  $(A - rI_2)\mathbf{v} = \mathbf{0}$ .
- (b) Use this idea to solve the system  $t\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}.$
- (c) Use this idea to solve the system  $t\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}.$

**Question 5.** For the system

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x},$$

do the following:

- (a) Find the general solution.
- (b) Now find the general solution using the fundamental matrix  $\Phi(t)$ , where  $\Phi(0) = I_2$ .
- (c) For each, solve for the particular solutions using the initial conditions  $\mathbf{x}(0) = \begin{bmatrix} m \\ 1 \end{bmatrix}$ ,  $m = 1, 2, 3$ .

**Question 6.** For the system  $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ , do the following:

- (a) Solve the system using the fundamental matrix  $\Phi(t)$ , so that  $\Phi(0) = I_2$ .
- (b) Show by means of matrix multiplication that for  $s, t \in \mathbb{R}$ , we have

$$\Phi(s)\Phi(t) = \Phi(s+t).$$

(This is a very important feature of differential equations; they can be viewed as transformations of phase space, and these transformations form a group under composition. We would say that these transformations, one for each  $t \in \mathbb{R}$ , form a *one-parameter group* of transformations of phase space, or that this ODE system forms an  $\mathbb{R}$ -action on the plane, in this instance.)