CHALLENGE PROBLEM SET: WEEK 9

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. Transform the system

$$\dot{x}_1 = -\frac{1}{2}x_1 + 2x_2, \quad x_1(0) = -2$$
$$\dot{x}_2 = -2x_1 - \frac{1}{2}x_2, \quad x_2(0) = 2$$

into a single equation of second order. Find the solution functions $x_1(t)$ and $x_2(t)$ that satisfy the initial conditions and sketch the solution curve in the x_1x_2-plane for $t > 0$.

Question 2. Let $x = x_1(t), y = y_1(t)$ and $x = x_2(t), y = y_2(t)$ be any two solutions to the ODE system

$$x' = p_{11}(t)x + p_{12}(t)y + g_1(t),$$
$$y' = p_{21}(t)x + p_{22}(t)y + g_2(t).$$

Show that $x = x_1(t) - x_2(t), y = y_1(t) - y_2(t)$ is then a solution to the corresponding homogeneous system.

Question 3. If $A(t) = \begin{bmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{bmatrix}$ and $B(t) = \begin{bmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{bmatrix}$, then calculate the following:

(a) $AB(t)$.

(b) $\dfrac{dA}{dt}$.

(c) $\displaystyle\int_0^1 B(t)\, dt$.

Question 4. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$.

Question 5. Prove that $\lambda = 0$ is an eigenvalue of a matrix A if and only if A is singular.

Question 6. Let $\mathbf{x}^{(1)}(t) = \begin{bmatrix} 2t \\ t^2 \end{bmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ be two 2-vector functions. Do the following:

- **(a)** Compute $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ and determine all intervals where the functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent.
- **(b)** Draw conclusions about the coefficients of the homogeneous system of equations satisfied by these two functions.
- **(c)** Find a homogeneous system of first order linear ODEs where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions.

Question 7. For the system

$$\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \mathbf{x},$$

find the general solution and sketch a phase portrait.

Question 8. Find the general solution to the system

$$\mathbf{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \mathbf{x}.$$