CHALLENGE PROBLEM SET: WEEK 7

110.302 DIFFERENTIAL EQUATIOMS
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Question 1. Use the Method of Undetermined Coefficients to solve the following:

(a) $y'' + y = t(1 + \sin t)$.

(b) $y'' = \frac{1}{2}\left(t^2 + 3\sin t - y - 2y'\right)$.

Question 2. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$, for $m = 1, \ldots, N$.

Question 3. In many mathematical models, the forcing function of a nonhomogeneous ODE may be piecewise defined and may not actually be continuous. As an example, consider the ODE

$$y'' + y = \begin{cases} t, & 0 \leq t \leq \pi \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assuming that y and y' are continuous at $t = \pi$, Solve the IVP and plot the forcing function and the solution as functions of time. Note that one should solve the IVP only on the interval $[0, \pi]$ first, and then simply create a new IVP using the other piece of the forcing function and a new set of initial conditions determined by the end of the solution on the interval $[0, \pi]$.

Question 4. Follow the instructions of the previous problem to solve the IVP with a discontinuous forcing function

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \leq t \leq \frac{\pi}{2} \\ 0, & t > \frac{\pi}{2}, \end{cases}$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

Question 5. Given the ODE $(1 - t)y'' + ty' - y = 2(t - 1)^2 e^{-t}$, defined on the interval $0 < t < 1$, do the following:

(a) Verify that both $y_1(t) = e^t$ and $y_2(t) = t$ solve the homogeneous version of the ODE, and use these and the method of Variation of Parameters to find a general solution.

(b) Now use the Reduction of Order method, a la Problem 3.6.28 in the book, using the single solution $y_1(t) = e^t$.

(c) Now use the same two previous methods on the ODE $x^2 y'' - 3xy' + 4y = x^2 \ln x$, for $x > 0$, knowing that both $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ solve the homogeneous version.