CHALLENGE PROBLEM SET: WEEK 6

110.302 DIFFERENTIAL EQUATIOMS
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Question 1. If the Wronskian determinant of two functions $f(x)$ and $g(x)$ is $3e^{4x}$, and if $g(x) = e^{2x}$, find $f(x)$.

Question 2. Do the following for the Initial Value Problem

$$y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

- **(a)** Find a solution $y(t)$.
- **(b)** Find α so that $y = 0$ when $t = 1$.
- **(c)** Find, as a function of α, the smallest positive value of t for which $y = 0$.
- **(d)** Determine the limit of the expression you found in part (c) as $t \to \infty$.

Question 3. Euler Equations An equation of the form

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0, \tag{1}$$

where α and β are real constants, is called an Euler Equation.

- **(a)** Let $x = \ln t$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
- **(b)** Use the results of part (a) to transform Equation 1 into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{2}$$

  Observe that Equation 2 has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions for Equation 2, then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions to Equation 1
- **(c)** Use this to find a fundamental set of solutions to $t^2y'' + 4ty' + 2y = 0$.
- **(d)** Use this to find a fundamental set of solutions to $t^2y'' - 3ty' + 4y = 0$.

Question 4. If the roots of a characteristic equation are real, show that a solution to $ay''+by'+cy = 0$ is either everywhere 0 or else can take on the value 0 at most once. (Note that there are different cases here.)

Question 5. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of turbulent flow of of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.