CHALLENGE PROBLEM SET: WEEK 4

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Do the following for the second-order linear homogeneous ODE

$$t^2y'' + 4ty' + 4y = 0.$$

(a) Determine two values r_1 and r_2 such that $y_1(t) = t^{r_1}$ and $y_2(t) = t^{r_2}$ each solve the ODE.

(b) Show that, for any two real constants $c_1, c_2 \in \mathbb{R}$, $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution (this is always true for linear homogeneous ODEs of any order and is called the Principle of Superposition).

(c) Find values for c_1 and c_2 so that $y(t) = c_1y_1(t) + c_2y_2(t)$ solves the IVP

$$t^2y'' + 4ty' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Question 2. For the first-order linear IVP

$$y' - \frac{3}{2}y = 3t - 2e^t, \quad y(0) = y_0,$$

find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. Then determine the longterm behavior of this solution.

Question 3. Solve the IVP

$$y' = \frac{x(x^2+1)}{4y^3}, \quad y(0) = -\frac{1}{\sqrt{2}}$$

by finding an implicitly defined solution. Then solve for the explicit solution $y(x)$, graph it and determine completely its domain.

Question 4. Solve the IVP $(4 - t^2)y' + 2ty = 2t^3$, $y(1) = -3$.

Question 5. Solve the ODE $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2 \cos x)dy = 0$.

Question 6. For the ODE $(x+2)\sin y + (x \cos y)y' = 0$, assume that you know that the integrating factor is only a function of x, so $\mu(x, y) = \mu(x)$. Following the discussion on page 99 of the text, find μ so that the ODE can be made exact and then solve the ODE.

Question 7. The Malthusian model for population growth is $y' = ry$, $y(0) = y_0 \geq 0$, with solutions $y(t) = y_0 e^{rt}$. This is the standard model exhibiting exponential growth and is a good model when populations are small and resources are unlimited or in great supply. However, with limited resources, it does not model large populations well. One way to generalize the model is to replace the constant growth rate r with a linear growth rate $r - ay$ that decays as populations rise, with r as before and $a > 0$. The new model is then

$$y' = (r - ay)y, \quad y(0) = y_0 \geq 0,$$

and is called the Verhulst Equation or the Logistic Equation, shown to be quite accurate in modeling populations. Do the following:

(a) Draw a phase line for this new model and determine the long term behavior for all populations y_0 in terms of a and r.

(b) Now solve the ODE for $a = 1$ and $r = 2$ explicitly (it is separable), writing the constant of integration in terms of y_0 and show that the long term behavior of any solution with a positive starting population y_0 is the same as the phase line for $a = 1$ and $r = 2$. You can use the calculations in the book, on page 82-3 to help you through these calculations.

(c) For $r = 2$ and $a = 1$, graph some representative solutions for different values for y_0.