DIFFERENTIABILITY IMPLIES CONTINUITY

AS.110.106 CALCULUS I (BIO & SOC SCI)
PROFESSOR RICHARD BROWN

Here is a theorem that we talked about in class, but never fully explored; the idea that any differentiable function is automatically continuous. We did offer a number of examples in class where we tried to calculate the derivative of a function at a point where the function was not continuous. All of these attempts failed. But we never actually showed why a function must be continuous to have a derivative.

Theorem 0.1. *If a function $f(x)$ is differentiable at a point $x = c$ in its domain, then $f(c)$ is continuous at $x = c$.*

Note that the converse is definitely not true, as, for example, $f(x) = |x|$ is continuous at $x = 0$, but not differentiable there. Note also that, for a function $f(x)$ to be continuous at $x = c$, we must have

$$\lim_{x \to c} f(x) = f(c).$$

But we can also write this as

$$\lim_{x \to c} f(x) - f(c) = 0.$$

This will be useful.

Proof. Assume we have a function $f(x)$ that is differentiable at a point $x = c$ in its domain. Then the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. Knowing this, we calculate $\lim_{x \to c} f(x) - f(c)$ as follows (we would like it to be 0):

$$\begin{aligned}
\lim_{x \to c} f(x) - f(c) &= \lim_{x \to c} \left(f(x) - f(c)\right)\left(\frac{x - c}{x - c}\right) && \text{Clever form of 1 multiplication} \\
&= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}\right)(x - c) && \text{Algebraic rearrangement} \\
&= \underbrace{\left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right)}_{f'(c)} \underbrace{\left(\lim_{x \to c} x - c\right)}_{0} && \text{Product Rule for limits} \\
&= f'(c) \cdot 0 = 0.
\end{aligned}$$

Hence $\lim_{x \to c} f(x) - f(c) = 0$, so that $\lim_{x \to c} f(x) = f(c)$ and $f(x)$ is continuous at $x = c$. □