CHALLENGE PROBLEM SET: CHAPTER 5, SECTIONS 1 AND 2, COURSE WEEK 9

110.201 LINEAR ALGEBRA
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Question 1. Do the following three problems:

(a) Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

(b) Determine whether the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}$ is acute, right, or obtuse.

(c) Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{in } \mathbb{R}^n.$$

For $n = 2, 3, 4$, find the angle θ between $\mathbf{u}$ and $\mathbf{v}$. For $n = 2, 3$, represent the vectors graphically. Then find the limit of θ as n approaches infinity.

Question 2. For the following situations, find a basis for the subspace $W^\perp \in \mathbb{R}^4$, where

$$\textbf{(a)} \quad W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right), \qquad \textbf{(b)} \quad W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right),$$

$$\textbf{(c)} \quad W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Question 3. For a line L in $\mathbb{R}^2$ passing through the origin, draw a sketch to interpret the following transformations graphically:

(a) $T(\mathbf{x}) = \mathbf{x} - \text{proj}_L \mathbf{x}$.

(b) $T(\mathbf{x}) = \mathbf{x} - 2\text{proj}_L \mathbf{x}$.

(c) $T(\mathbf{x}) = 2\text{proj}_L \mathbf{x} - \mathbf{x}$.

Question 4. Find scalars a, b, c, d, e, f, g so that the three vectors here are orthonormal:

$$\begin{bmatrix} a \\ d \\ f \end{bmatrix}, \quad \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \quad \begin{bmatrix} c \\ e \\ \frac{1}{2} \end{bmatrix}.$$

Question 5. Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ spanned by

$$\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Question 6. Find the orthogonal projection of $\mathbf{e}_1 \in \mathbb{R}^4$ onto the subspace spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Question 7. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^4$; We are told that $\mathbf{v}_i \cdot \mathbf{v}_j$ is the entry a_{ij} entry of

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 11 \\ 5 & 9 & 20 \\ 11 & 20 & 49 \end{bmatrix}.$$

Do the following:

(a) Find $||\mathbf{v}_2||$.

(b) Find the angle enclosed by vectors $\mathbf{v}_2$ and $\mathbf{v}_3$.

(c) Find $||\mathbf{v}_1 + \mathbf{v}_2||$.

(d) Find $\text{proj}_{\mathbf{v}_2}(\mathbf{v}_1)$ expressed as a scalar multiple of $\mathbf{v}_2$.

(e) Find a non-zero vector $\mathbf{v}$ in $\text{span}(\mathbf{v}_2, \mathbf{v}_3)$ such that $\mathbf{v}$ is orthogonal to $\mathbf{v}_3$. Express $\mathbf{v}$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

(f) Find $\text{proj}_V(\mathbf{v}_3)$, where $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Express your answer as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Question 8. Perform a Gram-Schmidt process on the sequences of vectors given:

$$\text{(a)} \quad \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix}.$$

Question 9. Perform a QR factorization on the matrices given:

$$\text{(a)} \quad \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 5 & 3 \\ 4 & 6 \\ 2 & 7 \\ 2 & -2 \end{bmatrix}.$$

Question 10. Find an orthonormal basis of:

$$\text{(a) The kernel of } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \qquad \text{(b) The image of } \mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$