CHALLENGE PROBLEM SET: CHAPTER 3, SECTION 4, COURSE WEEK 6

110.201 LINEAR ALGEBRA
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Question 1. For the following sets of vectors, verify that $\mathbf{x}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then find the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$. Write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

$$\text{(a)} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$\text{(b)} \quad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix},$$

$$\text{(c)} \quad \mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Question 2. In each instance (or working in three subgroups), the matrix $\mathbf{A}$ below is with respect to the standard basis. Find the matrix $\mathbf{B}$ of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the given basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$:

$$\text{(a)} \quad \mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\text{(b)} \quad \mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

$$\text{(c)} \quad \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}; \; \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

Question 3. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis of this plane $\mathcal{B}$ so that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

Question 4. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the $\mathcal{B}$-matrix of the linear transformation

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\mathbf{x} \quad \text{is} \quad B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

Question 5. Consider a basis $\mathcal{B}$ of $\mathbb{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\mathcal{R}$ be the basis consisting of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix $\mathbf{P}$ such that

$$[\mathbf{x}]_{\mathcal{R}} = \mathbf{P}\,[\mathbf{x}]_{\mathcal{B}}\,.$$

Question 6. Do the following:

(a) If $c \neq 0$, find the matrix of the linear transformation $T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x}$ of $\mathbb{R}^2$ with respect to the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix}$.

(b) Find an invertible 2×2 matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\mathbf{S}$$

is of the form $\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$.

Question 7. For the matrix $\mathbf{A} = \begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}$, find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the $\mathcal{B}$-matrix $\mathbf{B}$ of $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Question 8. Questions to argue over:

(a) If $\mathbf{A}$ is a 5×6 matrix of rank 4, then the nullity of $\mathbf{A}$ is 1.

(b) The identity matrix is similar to ALL $n \times n$ invertible matrices.

(c) The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$, where a and b are arbitrary real numbers, form a subspace of $\mathbb{R}^4$.

(d) If $\mathbf{A}$ is an invertible matrix, then the kernels of $\mathbf{A}$ and $\mathbf{A}^{-1}$ must be equal.

(e) There exists a 2×2 matrix $\mathbf{A}$ where $\text{im}(\mathbf{A}) = \ker(\mathbf{A})$.

(f) If $\mathbf{A}$ is similar to $\mathbf{B}$, then there exists *one and only one* invertible matrix $\mathbf{S}$, such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$.

(g) If the image of an $n \times n$ matrix $\mathbf{A}$ is all of $\mathbb{R}^n$, then $\mathbf{A}$ must be invertible.