CHALLENGE PROBLEM SET: CHAPTER 2, COURSE WEEK 4

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$. Do the following:

(a) Find all values of k so that $\mathbf{A}$ is invertible.

(b) Find all values of k so that all of the entries of $\mathbf{A}^{-1}$ are integers.

Question 2. Do the following:

(a) For $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, determine if the transformation $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ from $\mathbb{R}^3$ to $\mathbb{R}$ is linear. If it is, then find the matrix for T.

(b) Do the same for the arbitrary vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

(c) Conversely, consider an arbitrary linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$. Show that it can always be written as $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ for some choice of vector $\mathbf{v} \in \mathbb{R}^3$.

Question 3. Let L be a line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. FInd the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

Question 4. Given a reflection matrix $\mathbf{A}$ and a vector $\mathbf{x} \in \mathbb{R}^2$, define $\mathbf{v} = \mathbf{x} + \mathbf{A}\mathbf{x}$ and $\mathbf{w} = \mathbf{x} - \mathbf{A}\mathbf{x}$.

(a) Using the definition of a reflection, find $\mathbf{A}(\mathbf{A}\mathbf{x})$ in terms of $\mathbf{x}$.

(b) Express $\mathbf{A}\mathbf{v}$ in terms of $\mathbf{v}$.

(c) Express $\mathbf{A}\mathbf{w}$ in terms of $\mathbf{w}$.

(d) If the vectors $\mathbf{v}$ and $\mathbf{w}$ are both non-zero, then what is the angle between $\mathbf{v}$ and $\mathbf{w}$?

(e) If $\mathbf{v}$ is non-zero, then what is the relationship between $\mathbf{v}$ and the line of reflection L?

Question 5. Find all matrices that commute with the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Question 6. Divide your group into three groups and work the following problem, each with one different matrix $\mathbf{A}$ below, and parallel on a board. Present your solutions in turn and contrast your solution with the others in this group. Discuss. For each matrix $\mathbf{A}$ given, calculate the first few "powers" of $\mathbf{A}$. This means $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$, and so on. Then use the pattern given to find $\mathbf{A}^{1001}$. Interpret your answers geometrically, in terms of compositions of reflections, rotations, scalings, orthogonal projections and shears.

$$\text{(a)}\quad \mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \text{(b)}\quad \mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{(c)}\quad \mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Question 7. Now divide your group into two groups and work the following problem, each with one different matrix from (a) and (b). Find ALL matrices $\mathbf{A}$ that satisfy the given matrix equation:

$$\text{(a)}\quad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\mathbf{A} = \mathbf{I}_3, \quad \text{(b)}\quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\mathbf{A} = \mathbf{I}_2.$$

Question 8. Which of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are invertible. For those that are, describe in detail the inverse transformation.

(a) Reflection about a plane.

(b) orthogonal projection onto a plane.

(c) Scaling by a factor of 5 (That is, $T(\mathbf{x}) = 5\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^3$.)

(d) Rotation about an axis.

Question 9. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Fully explain your answer.

Question 10. Consider an invertible linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(\mathbf{x}) = \mathbf{Ax} = \mathbf{y}$ with inverse linear transformation $L = T^{-1}$, from $\mathbb{R}^n$ to $\mathbb{R}^m$. Since L is also linear, we know there is a $m \times n$ matrix $\mathbf{B}$, where $L(\mathbf{y}) = \mathbf{By} = \mathbf{x}$. Use the equations $\mathbf{BA} = \mathbf{I}_m$ and $\mathbf{AB} = \mathbf{I}_n$ to show that m must equal n. (As a hint, think about the number of solutions to the linear systems $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{Bx} = \mathbf{0}$.)

Question 11. Do the following:

(a) Consider an $n \times m$ matrix $\mathbf{A}$ with rank($\mathbf{A}$)$< n$. Show that there exists a vector $\mathbf{b} \in \mathbb{R}^n$ such that the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent. Hint: For $\mathbf{E}$ = rref($\mathbf{A}$), show that there exists a vector $\mathbf{c} \in \mathbb{R}^n$ such that the system $\mathbf{Ex} = \mathbf{c}$ is inconsistent. Than "work backward".

(b) Let $\mathbf{A}$ be $n \times m$, with $n > m$. Show that there exists a vector $\mathbf{b} \in \mathbb{R}^n$ such that the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent.