CHALLENGE PROBLEM SET 11: CHAPTER 7, SECTIONS 3 AND 5, COURSE WEEK 13

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. For the following matrices, find all eigenvalues. If possible, find an eigenbasis for each:

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, **(c)** $\begin{bmatrix} 3 & 0 & -2 \\ -3 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$, **(d)** $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

Question 2. Find all eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is there an eigenbasis? Interpret your answer geometrically.

Question 3. What can you say about the geometric multiplicity of the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix},$$

where a, b, and c are arbitrary constants?

Question 4. Do the following:

(a) For a diagonal matrix $\mathbf{A}_{n\times n}$, where $\text{rank}(\mathbf{A}) = r < n$, find the algebraic and geometric multiplicities of the 0-eigenvalue in terms of r and n.

(b) For an upper-triangular matrix $\mathbf{A}_{n\times n}$, where $a_{ii} \neq 0$, for $i = 1, \ldots, m$, and $a_{ii} = 0$, for $i = m+1, \ldots n$, find the algebraic multiplicity of the 0-eigenvalue of $\mathbf{A}$. Without using Theorem 7.3.6, what can you say about the geometric multiplicity?

(c) Suppose $\mathbf{A}_{n\times n}$ has an eigenbasis. What is the relationship between the algebraic and geometric multiplicities of the eigenvalues of $\mathbf{A}$?

Question 5. For which values of the constants a, b, and c are the following matrices diagonalizable?

(a) $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$, **(b)** $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$, **(c)** $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Question 6. Do the following:

(a) Write $z = 3 - 3i$ in polar form.

(b) Show for an non-zero complex number $z \in \mathbb{C}$, there are exactly two complex numbers w such that $w^2 = z$. If z is in polar form, describe w in polar form also.

(c) Describe the linear transformation $T : \mathbb{C} \to \mathbb{C}$, $T(z) = (1 - i)z$ geometrically.

(d) Consider a polynomial $f(\lambda)$ with real coefficients. Show if $a + bi \in \mathbb{C}$ is a root of $f(\lambda)$, then so is its complex conjugate $a - bi$.

Question 7. For the following matrices, find a matrix $\mathbf{S}$ so that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $a, b \in \mathbb{R}$.

$$\textbf{(a)} \quad \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}, \qquad \textbf{(b)} \quad \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}.$$

Question 8. For the following matrices, find all eigenvalues (real and complex):

$$\textbf{(a)} \quad \begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}, \qquad \textbf{(b)} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \textbf{(c)} \quad \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Question 9. Do the following:

(a) Suppose $\mathbf{A}_{3\times 3}$ has only two distinct eigenvalues. If trace($\mathbf{A}$) $= 1$ and det($\mathbf{A}$) $= 3$, find the eigenvalues of $\mathbf{A}$ with their algebraic multiplicities.

(b) Consider

$$\mathbf{A} = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$ are all positive. If $\mathbf{A}$ has distinct eigenvalues, what can you say about the signs of the eigenvalues (how many are positive, negative or zero?) What is the sign of the largest (in magnitude) eigenvalue?.

Question 10. Given

$$\mathbf{J}_n(k) = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ 0 & 0 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix},$$

where $\mathbf{J}_n(k)$ has k's along the main diagonal (k is an arbitrary real constant), and 1's directly above in each column, find the eigenvalues of $\mathbf{J}_n(k)$ with their algebraic and geometric multiplicities.