CHALLENGE PROBLEM SET: CHAPTER 6, SECTION 3, CHAPTER 7, SECTIONS 1 AND 2, COURSE WEEK 12

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. Use the geometric interpretation of the determinant of a 2×2 matrix (from Section 6.3) to find the areas of the following regions:

(a) The parallelogram defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

(b) The triangle in the figure at left below.

(c) The region in the figure at right below.

Question 2. For $\mathbf{v}$ an eigenvector of both $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$, determine the following:

(a) Is $\mathbf{v}$ necessarily an eigenvector of $\mathbf{A}+\mathbf{B}$?

(b) Is $\mathbf{v}$ necessarily an eigenvector of $\mathbf{AB}$?

In each case, if so, then determine the corresponding eigenvalue for $\mathbf{v}$.

Question 3. Do the following:

(a) Show that 4 is an eigenvalue of $\mathbf{A} = \begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix}$ and find all eigenvectors.

(b) Find all 2×2 matrices that have $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as an eigenvector with eigenvalue -1.

(c) Find all 2×2 matrices that have $\mathbf{e}_1$ as an eigenvector.

Question 4. Show that similar matrices have the same eigenvalues. *Hint:* If $\mathbf{v}$ is an eigenvalue of $\mathbf{S}^{-1}\mathbf{AS}$, then $\mathbf{Sv}$ is an eigenvector of $\mathbf{A}$.

Question 5. Find a 2×2 matrix where $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors with respective eigenvalues 5 and 10.

Question 6. For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, do the following:

(a) Use the geometric interpretation of $\mathbf{A}$ as a reflection combined with a scaling to find the eigenvalues of $\mathbf{A}$.

(b) Find an eigenbasis of $\mathbf{A}$.

(c) Diagonalize $\mathbf{A}$.

Question 7. Let V be the linear space of all 2×2 matrices for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

(a) Find a basis for V and determine its dimension.

(b) Consider the linear transformation $T : V \to \mathbb{R}$, $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find image($T$) and ker($T$). What is the rank of T?

(c) Consider the linear transformation $T : V \to \mathbb{R}$, $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find image($T$) and ker($T$). What is the rank of T?

Question 8. For the following matrices, find all eigenvalues and determine their algebraic multiplicities:

(a) $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$, **(b)** $\begin{bmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{bmatrix}$, **(c)** $\begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$.

Question 9. Consider the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where a, b and c are non-zero constants. For which values of a, b, and c does $\mathbf{A}$ have two distinct eigenvalues?

Question 10. Consider the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where a, and b are arbitrary constants. Find all eigenvectors of $\mathbf{A}$.