EXAMPLE: CHAPTER 2 HOMEWORK PROBLEM

110.201 LINEAR ALGEBRA
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Problem (2.1.13a)**.** Prove the following statement: The 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$.

Strategy. This is a biconditional statement, so we have to prove the conditional statement in each direction. To do this, we will write out the system of equations $\mathbf{Ax} = \mathbf{y}$, and use elementary row operations to change the system to an equivalent one where the idea of $\mathbf{A}$ being invertible is equivalent to the statement that $ad - bc \neq 0$.

Solution. We first prove the statement: If $\mathbf{A}$ is invertible, then $ad - bc \neq 0$. We prove this by proving the contrapositive: If $ad - bc = 0$, then $\mathbf{A}$ is not invertible. Suppose $ad - bc = 0$. The matrix equation $\mathbf{Ax} = \mathbf{y}$ corresponds to the system

$$\begin{aligned} (I) \quad & ax_1 + bx_2 = y_1 \\ (II) \quad & cx_1 + dx_2 = y_2, \end{aligned}$$

where we have conveniently named the equations to refer to them. By replacing equation (II) by the new equation $(III) = d(I) - b(II)$, we obtain a system formed by equations (I) and (III) with the same solutions. We get

$$\begin{aligned} (I) \quad & ax_1 + bx_2 = y_1 \\ (III) \quad & (ad - bc)x_1 + 0x_2 = dy_1 - by_2. \end{aligned}$$

Since we are assuming that $ad - bc = 0$, the last equation in this new system reduces to $0 = dy_1 - by_2$. This must be true for each and every choice of $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. For any particular choice of $\mathbf{y}$, we will have either $dy_1 = by_2$, and the system will have many solutions, or $dy_1 \neq by_2$, and the system will have no solutions (it will be inconsistent). In either case, there will be no inverse linear transformation, and the matrix $\mathbf{A}$ will not be invertible.
Now we prove the statement: If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible. Given the same system as above, with the same elementary row operation, we now see that the second equation of the system

$$(ad - bc)x_2 = dy_1 - cy_2 \quad \Longrightarrow \quad x_2 = \frac{dy_1 - by_2}{ad - bc} = \left(\frac{d}{ad - bc} \right) y_1 + \left(\frac{-b}{ad - bc} \right) y_2$$

will have a unique solution given any choice of y_1 and y_2 and knowing that $ad - bc \neq 0$. Thus there will be a unique solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for any choice of $\mathbf{y}$. This means that $\mathbf{A}$ will be invertible (we will be able to produce the backwards function).

And as a special note, finishing the elementary row reductions on the system we played with above will produce exactly the structure of the matrix $\mathbf{A}^{-1}$ needed for Problem 1.1.13b.