Solution to Problem 1.1.18

1.1 18) $\begin{bmatrix} x+2y+3z=a \\ x+3y+8z=b \\ x+2y+2z=c \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 3 & 8 & b \\ 1 & 2 & 2 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & b-a \\ 0 & 0 & -1 & c-a \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & \begin{matrix} a-2b-2a \\ 3a-2b \end{matrix} \\ 0 & 1 & 5 & b-a \\ 0 & 0 & 1 & c-a \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \begin{matrix} 3a-2b+7(c-a) \\ 10a-2b-7c \end{matrix} \\ 0 & 1 & 0 & \begin{matrix} b-a-5a+5c \\ -6a+b+5c \end{matrix} \\ 0 & 0 & 1 & a-c \end{array}\right]$

$$\boxed{\begin{aligned} x &= 10a-2b-7c \\ y &= -6a+b+5c \\ z &= a-c \end{aligned}}$$

Describe for me the value of this "solution" in terms of

1. The way it answers (or addresses) the question.
2. Its correctness.
3. Its ability for use as a tool
   - (a) for study
   - (b) as a template
   - (c) as an organizational model.

Another solution to Problem 1.1.18

1.1 18) Here we have 3 equations and 3 variables. Our goal, through multiplying, subtracting and adding the rows will be to isolate each variable.

$$\begin{array}{l|l|l} (\mathrm{I}) & x + 2y + 3z = a & \\ (\mathrm{II}) & x + 3y + 8z = b & (-\mathrm{I}+\mathrm{II}) \\ (\mathrm{III}) & x + 2y + 2z = c & (-\mathrm{I}+\mathrm{III}) \end{array}$$

$$\Rightarrow \left|\begin{array}{l} x + 2y + 3z = a \\ y + 5z = -a+b \\ -z = -a+c \end{array}\right| \Rightarrow \left|\begin{array}{l} x + 2y + 3z = a \\ y + 5z = b-a \\ z = a-c \end{array}\right| \begin{array}{l} (-3\,\mathrm{III}+\mathrm{I}) \\ (-5\,\mathrm{III}+\mathrm{II}) \\ \end{array}$$

$$\Rightarrow \left|\begin{array}{l} x + 2y = 3c-2a \\ y = 5c+b-6a \\ z = a-c \end{array}\right| \overset{(\mathrm{I}-2\mathrm{II})}{\Rightarrow} \left|\begin{array}{l} x = 3c-2a-2(5c+b-6a) \\ y = 5c+b-6a \\ z = a-c \end{array}\right|$$

$$\vdots$$

$$\Rightarrow \left|\begin{array}{l} x = 10a-2b-7c \\ y = -6a+b+5c \\ z = a-c \end{array}\right|$$

- From this linear system, we see that the planes defined by the 3 equations intersect at the pt $(x,y,z) = (10a-2b-7c, -6a+b+5c, a-c)$

- These three planes intersect at this unique pt.

Yet another solution to 1.1.18

1.1.18) Find all solutions to the linear system

$$\left| \begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array} \right|$$

where a, b, c are arbitrary constants.

<u>Strategy</u>: By creatively replacing eqns here with sums of multiples of other equations, we can create a system with the same solutions but with equations easier to solve. We seek to create a system where each equation only contains 1 variable each of $\{x, y, z\}$.

<u>Solution</u> Label the equations

$$\begin{array}{ll} (\mathrm{I}) & x + 2y + 3z = a \\ (\mathrm{II}) & x + 3y + 8z = b \\ (\mathrm{III}) & x + 2y + 2z = c \end{array}$$

Using book notation, we have

$$\left| \begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array} \right| \begin{array}{l} \\ -(\mathrm{I}) + (\mathrm{II}) \\ -(\mathrm{I}) + (\mathrm{III}) \end{array} \longrightarrow \left| \begin{array}{r} x + 2y + 3z = a \\ y + 5z = b - a \\ -z = c - a \end{array} \right|$$

1.1, 18) cont'd.

And

$$\left|\begin{aligned} x+2y+3z &= a \\ y+5z &= b-a \\ z &= a-c \end{aligned}\right| \begin{matrix} -3(\text{III})+(\text{I}) \\ -5(\text{III})+(\text{II}) \end{matrix} \longrightarrow \left|\begin{aligned} x+2y \quad &= 3c-2a \\ y \quad &= -6a+b+5c \\ z &= a-c \end{aligned}\right|$$

And

$$\left|\begin{aligned} x+2y \quad &= 3c-2a \\ y \quad &= -6a+b+5c \\ z &= a-c \end{aligned}\right| \begin{matrix} -2(\text{II})+(\text{I}) \\ \longrightarrow \end{matrix} \left|\begin{aligned} x \quad &= 10a-2b-7c \\ y &= -6a+b+5c \\ z &= a-c \end{aligned}\right|$$

For any choice of constants a, b, c, the set of all solutions to the system is given by

$$\left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{aligned} x &= 10a-2b-7c \\ y &= -6a+b+5c \\ z &= a-c \end{aligned} \right\}$$