④ $V = \{ \underset{\text{in } P_2}{p(t)} \mid \int_0^1 p(t) = 0 \}$. If $p(x) \in V$, then $p(x) = a + bx + cx^2$ for some a, b, c.

Let's check to see if V is a subspace of the space of 2nd degree polynomials (P_2).

1) Does V contain a null element? (Yes)

Yes! If $p(t) = 0$, then $\int_0^1 p(t)\,dt = \int_0^1 dt\,[0] = 0$.

This implies that $0 \in V$.

2) Is V closed under addition? (Yes)

If $p(t), q(t) \in V$, then $\int_0^1 p(t)\,dt = 0$ and $\int_0^1 q(t)\,dt = 0$ by definition of V.

Then $p(t) + q(t) \equiv w(t)$, and $\int_0^1 w(t)\,dt = \int_0^1 (p(t) + q(t))\,dt = \int_0^1 p(t)\,dt + \int_0^1 q(t)\,dt$
$= 0$

Thus, $p(t) + q(t) \equiv w(t) \in V$, so V is closed under addition.

3) ~~Does~~ Is V closed under scalar multiplication? (Yes)

If k is a constant and $p(t) \in V$, then $\int_0^1 k\,p(t)\,dt = k \int_0^1 p(t)\,dt = k \cdot 0 = 0$.

V is a subspace of P_2 because it contains a null element, is closed under addition, and is closed under scalar multiplication.

Let's find a basis for V. To do this, we need to know what the elements of V look like!

$$\int_0^1 p(t)\,dt = 0 \implies \int_0^1 (a + bt + ct^2)\,dt = 0 \implies a + \tfrac{1}{2}b + \tfrac{1}{3}c = 0$$

We can "eliminate" one of these constants by solving for it in terms of the other two, so let $a = -\frac{1}{2}b - \frac{1}{3}c$. This will always hold in V because any arbitrary $p(t)$ in V has the form $a + bt + ct^2$ and obeys $\int_0^1 p(t)\,dt = 0$. Thus, we know that $p(t) = -\frac{1}{2}b - \frac{1}{3}c + bt + ct^2 = b(t - \frac{1}{2}) + c(t^2 - \frac{1}{3})$.

Thus, any polynomial $p(t)$ is a linear combination of the polynomials $t - \frac{1}{2}$ and $t^2 - \frac{1}{3}$ iff $p(t) \in V$, so the basis for V is $\{(t - \frac{1}{2}), (t^2 - \frac{1}{3})\}$

(10) Consider $V = \{ A \in \mathbb{R}^{3\times3} \mid \begin{pmatrix}1\\2\\3\end{pmatrix} = \vec{v} \in \ker A \}$. Note, $\mathbb{R}^{3\times3}$ means ~~[illegible]~~ the space of 3×3 matrices with real entries, and any element of the set V is a 3×3 matrix A satisfying the condition $A\vec{v} = 0$ for $\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Note, this means that $\vec{v}$ is <u>not</u> an element of V, and no vector in $\mathbb{R}^3$ is an element of V, by definition of V. Let's check to see if V is a subspace (it is)

1) The zero matrix $A = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ is an element of V because $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, thus $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \ker A$, and because A (the zero matrix) satisfies this property, it <u>must</u> be an element of V, by definition.

2) If $A, B \in V$, then $A\vec{v} = 0$ and $B\vec{v} = 0$. Thus, $(A+B)\vec{v} = A\vec{v} + B\vec{v} = 0$
$\Longrightarrow A+B \in V$.

3) If k is a constant and $A \in V$, then $(kA)\vec{v} = k(A\vec{v}) = k \cdot 0 = 0$.

V satisfies the three properties of subspaces, so V is a subspace. ~~[illegible]~~

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(18) We must find a basis for P_n and determine its dimension. $P_n = \{ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \text{ are constants} \}$

We have no restrictions on the polynomials in P_n, so none of the a_i necessarily depend on each other. Therefore, any $p(x) \in P_n$ looks like $p(x) = a_0(1) + a_1(x) + \cdots + a_n(x^n)$, so a natural basis for P_n is $\{1, x, x^2, \ldots, x^n\}$. There are $n+1$ elements in P_n because we have the element 1 and n powers of x in the basis, so the dimension of P_n is **$n+1$**.

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(24) $V = \{ A \in \mathbb{R}^{3\times3} \mid A \text{ is upper triangular} \}$. Any element of A takes the form

$$A = \begin{pmatrix}a&b&c\\0&d&e\\0&0&f\end{pmatrix}; \quad a,b,c,d,e,f \in \mathbb{R}.$$

We can break any matrix A into a linear combination of ~~matrix~~ basis matrices:

$$A = a\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + b\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} + c\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} + d\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + e\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} + f\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

Therefore, $\left\{ \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \right\}$ is a basis for V and because there are 6 basis elements, $\dim V = 6$.

(32) $V = \{A \in \mathbb{R}^{2\times 2} \mid \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A = A \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\}$

Let $A \in V$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We must find all A ~~that~~ that satisfy the constraint.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 2c & 0 \end{pmatrix}$$

$$\Rightarrow b+d=0 \Rightarrow b=-d$$

$$a+c=2a=2c \Rightarrow a=c$$

Thus, if $A \in V$, $A = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

Therefore, our basis is $\left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right\}$ because these matrices are linearly independent, and $\dim V = 2$.