(28) $v_1 = \begin{pmatrix}1\\0\\0\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\\3\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\0\\1\\4\end{pmatrix}$, $v_4 = \begin{pmatrix}2\\3\\4\\k\end{pmatrix}$

If $v_4$ is linearly independent from $v_1, v_2, v_3$, then

$$RREF\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} = I_{4\times4}.$$

$$\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\2&3&4&k\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&k-29\end{pmatrix}$$

if $k=29$, $RREF\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} \neq I_4$; else, we can reduce the matrix to the identity. Thus, $v_1, v_2, v_3, v_4$ form a basis for $\mathbb{R}^4$ if $k \neq 29$.

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(32) Any vector $\vec{v} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$ perpendicular to $\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$ must be in the kernel of the matrix $\begin{pmatrix}1&0&-1&1\\0&1&2&3\end{pmatrix}$ because $\vec{v}\cdot\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} = 0$ and $\vec{v}\cdot\begin{pmatrix}0\\1\\2\\3\end{pmatrix} = 0$ by the definition of perpendicular. Therefore, we are trying to solve $\begin{pmatrix}1&0&-1&1\\0&1&2&3\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. In augmented matrix form:

$$\left(\begin{array}{cccc|c}1&0&-1&1&0\\0&1&2&3&0\end{array}\right) \Rightarrow \quad a = c-d \quad \text{and} \quad b = -2c-3d.$$

So any vector $\vec{v}$ perpendicular to $\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$ has the form

$$\vec{v} = \begin{pmatrix}c-d\\-2c-3d\\c\\d\end{pmatrix} = c\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + d\begin{pmatrix}-1\\-3\\0\\1\end{pmatrix} \Rightarrow \left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-3\\0\\1\end{pmatrix}\right\}$$ is

a basis for the subspace of vectors in $\mathbb{R}^4$ that are perpendicular to $\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$

(10) $\left(\begin{array}{cc|c} 8 & 5 & 1 \\ 4 & 2 & -2 \\ -1 & -1 & -2 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 8 & 5 & 1 \\ -8 & -4 & 4 \\ -1 & -1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 8 & 5 & 1 \\ 0 & 1 & 5 \\ -8 & -8 & -16 \end{array}\right) \longrightarrow \left(\begin{array}{ccc} 8 & 5 & 1 \\ 0 & 1 & 5 \\ 0 & -3 & -15 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 8 & 5 & 1 \\ 0 & -5 & -25 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 8 & 0 & -24 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right)$

The system of equations is consistent and in the basis

$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix}}_{= v_1}, \underbrace{\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}}_{= v_2} \right\}$, $\vec{x} = -3v_1 + 5v_2$, so $\vec{x} = \left[\begin{pmatrix} -3 \\ 5 \end{pmatrix}\right]_{\mathcal{B}}$

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(44) In the basis $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\}$, $\vec{x} = \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]_{\mathcal{B}}$.

Therefore, $\vec{x} = 2\begin{pmatrix} 8 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ -1 \end{pmatrix}$.

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(69) We know that $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We can choose a new basis $\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{= v_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{= v_2} \right\}$ so that a matrix $B$ is similar to $A$ by the relation $[A\vec{v}_{1,2}]_{\mathcal{B}} = B[\vec{v}_{1,2}]_{\mathcal{B}}$ where

$B = \left[ [A\vec{v}_1]_{\mathcal{B}} \ [A\vec{v}_2]_{\mathcal{B}} \right]$ (Thm 3.4.3), and $A = SBS^{-1}$ where

$S = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ (Thm 3.4.4). $[A\vec{v}_1]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $[A\vec{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow A$ is similar to $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$