Hw #4 Solutions

~~Describe~~

20) Describe the image of $A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{pmatrix}$ geometrically.

Let's begin to find RREF(A):

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{pmatrix} \xrightarrow[\text{Row II} = 2\times\text{Row II}]{\text{Row I} = 3\times\text{Row I}} \begin{pmatrix} 6 & 3 & 9 \\ 6 & 8 & 4 \\ 6 & 5 & 7 \end{pmatrix} \xrightarrow[R\,III = R\,I - R\,III]{R\,II = R\,I - R\,II} \begin{pmatrix} 6 & 3 & 9 \\ 0 & -5 & 5 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{RREF}(A)$$

$\Rightarrow \text{Rank}(A) = 2 \Rightarrow$ 2 vectors span $\text{Im}(A) \Rightarrow A$ is a plane in $\mathbb{R}^3$.

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30) Give an example of a matrix such that $\text{Im}(A)$ is spanned by $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

The column vectors of A must be in $\text{span}\{\begin{pmatrix} 1 \\ 5 \end{pmatrix}\}$. A must be a $2 \times n$ matrix. eg. $A = \begin{pmatrix} 1 & 2 \\ 5 & 10 \end{pmatrix}$ or

$A = \begin{pmatrix} 1 & 3 & -2 \\ 5 & 15 & -10 \end{pmatrix}$.

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6) (a) Is $V \cap W$ a subspace of $\mathbb{R}^n$ if V, W are subspaces of $\mathbb{R}^n$?

Yes! If $\vec{v}, \vec{w} \in V$ and W, then because V and W are closed under addition and scalar multiplication, $a\vec{v} + b\vec{w} \in V$ and W $\Longrightarrow a\vec{v} + b\vec{w} \in V \cap W$. Furthermore, $\vec{0} \in V$ and $\vec{0} \in W \Rightarrow \vec{0} \in V \cap W$

$\Rightarrow V \cap W$ is a subspace.

(b) Not necessarily. If $v \in V$ but $v \notin W$, and $w \in W$ but $w \notin V$, then it is not true that $v + w \in W$ or V necessarily.

Example: In $\mathbb{R}^2$, $V = \text{span}\{e_1\}$, $W = \text{span}\{e_2\}$. $e_1 + e_2 \notin V \cup W$.

(32) What is a basis of the image of $\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$?

Recall Thm 3.2.4: a basis for $\mathrm{Im}(A)$ is the columns of A containing pivots (ie, redundant vectors are not part of the basis). Therefore,

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for $\mathrm{Im}(A)$.

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(46) Find a basis for ker A where $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{pmatrix}$.

A basis for A may be found by finding nontrivial solutions to the equations $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 + e\vec{v}_5 = 0$, where the v_i are the column vectors of A.

We can do this by looking at each redundant vector:

$2v_1 = v_2 \implies 2v_1 - v_2 = 0 \implies z_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \ker A$

$3v_1 + 4v_3 = v_4 \implies 3v_1 + 4v_3 - v_4 = 0 \implies z_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \\ 0 \end{pmatrix} \in \ker A$

This is linearly independent from z_1.

$5v_1 + 6v_3 = v_5 \implies 5v_1 + 6v_3 - v_5 = 0 \implies z_3 = \begin{pmatrix} 5 \\ 6 \\ 6 \\ 0 \\ -1 \end{pmatrix} \in \ker A$.

This is linearly independent from z_1, z_2.

We have exhausted all redundant vectors $\implies z_1, z_2, z_3$ form a basis for $\ker A$.