HW #3 - Solutions

23) Which matrices commute with $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} a+2b & 3a+6b \\ c+2d & 3c+6d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+3c & b+3d \\ 2a+6c & 2b+6d \end{pmatrix}$$

We can set up a system of equations:

$$\begin{pmatrix} 0 & 2 & -3 & 0 \\ 2 & 0 & 5 & -2 \\ 3 & 5 & 0 & -3 \\ 0 & 2 & -3 & 0 \end{pmatrix} \xrightarrow[R_1 = R_1/2]{R4 = R_1 - R_4} \begin{pmatrix} 0 & 1 & -3/2 & 0 \\ 2 & 0 & 5 & -2 \\ 3 & 5 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_3 \\ R_3 \to R_1}]{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 5 & 0 & -3 \\ 0 & 1 & -3/2 & 0 \\ 2 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 = -2R_3]{R_1 = 2R_1} \begin{pmatrix} 6 & 10 & 0 & -6 \\ 0 & 1 & -3/2 & 0 \\ -6 & 0 & -15 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_1 = R_1/2]{R_3 = \frac{R_1 + R_3}{10}} \begin{pmatrix} 3 & 5 & 0 & -3 \\ 0 & 1 & -3/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 = R_2 - R_3} \begin{pmatrix} 3 & 5 & 0 & -3 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $t \equiv b = (3/2)c \implies c = \frac{2}{3}t$

Let $s \equiv a$ and $3a + 5b - 3d = 0$

$\implies 3s + 5t - 3d = 0 \implies d = s + \left(\frac{5}{3}\right)t$

$$\implies A = \begin{pmatrix} s & t \\ \frac{2}{3}t & s + \frac{5}{3}t \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

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29) a) Geometrically, D_α represents a counterclockwise rotation through an angle α. $D_\alpha D_\beta$ first rotates a vector by angle β C.C. and then by α C.C.. We expect that this should just be a rotation through angle $\beta + \alpha$. $D_\beta D_\alpha$ rotates through α first, then through β. Again, this should be a total rotation through $\alpha + \beta$. Thus, $D_\alpha D_\beta \vec{x} = D_\beta D_\alpha \vec{x}$ is what we should expect.

b) $* = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -(\sin\beta\cos\alpha + \cos\beta\sin\alpha) \\ \cos\beta\sin\alpha + \sin\beta\cos\alpha & \cos\beta\cos\alpha - \sin\beta\sin\alpha \end{pmatrix}$

Because $\cos(\beta)\cos(\alpha) \mp \sin(\beta)\sin(\alpha) = \cos(\beta \pm \alpha)$
and $\sin(\beta)\cos(\alpha) \pm \sin(\alpha)\cos(\beta) = \sin(\beta \pm \alpha)$
$\implies * = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$

Same for the other way.

62) Find all matrices X such that

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} X = \mathbb{1}_{3\times 3}$$

Two easy ways:

1) An $n\times m$ matrix A won't have a "right inverse" (ie, a matrix X such that $AX = \mathbb{1}_{n\times n}$) if $m<n$.

2) Set up a system of equations:

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3/2 & 0 & 1/2 \end{array}\right)$$

This system is inconsistent $\Longrightarrow$ no solutions.

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17) Is $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ invertible?

Compute $\det A = 1\times 8 - 2\times 4 = 8 - 8 = 0$

$\Rightarrow$ A is not invertible

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32) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ because $ad-bc=1$.

$b=-b$ and $c=-c \Longrightarrow b=c=0$.

$a=d$ and $d=a \Longrightarrow$ the matrix looks like $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \equiv A$

$\det A = a^2 = 1 \Rightarrow a = \pm 1$

$\Rightarrow$ all invertible 2×2 matrices with $ad-bc = \det A$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$