Last homework solutions
(to graded problems)

Problem 7.3.16 - Determine the eigenvalues and eigenspaces of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ and diagonalize A if possible.

Strategy: We solve the characteristic equation of A for its eigenvalues, $A\vec{v} = \lambda\vec{v}$ for $\vec{v}$ the eigenvectors corr to each eigenvalue, and if the geometric multiplicities equal the algebraic multiplicities of each eigenvalue, we diagonalize via the change of basis matrix S.

Problem 7.3.16 (cont'd)

Solution: Here the char eqn of A is

$$\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & -1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ -1-\lambda & -1 \end{vmatrix}$$

$$= (1-\lambda)\left(-\lambda(-1-\lambda)-(-2)\right)+2(-1)$$

$$= (1-\lambda)(\lambda^2+\lambda+2)-2$$

$$= \lambda^2+\lambda+2-\lambda^3-\lambda^2-2\lambda-2$$

$$= -\lambda^3 - \lambda = -\lambda(\lambda^2+1) = 0$$

This eqn has only one real solution, $\lambda = 0$.

Given $\lambda=0$, we solve $A\vec{v} = 0\vec{v} = \vec{0}$:

$$\left.\begin{aligned} v_1+v_2 &= 0 \\ -v_2-v_3 &= 0 \\ 2v_1+2v_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} v_1 &= -v_2 \\ v_2 &= -v_3 \end{aligned}\right\} \quad \begin{aligned} &\text{choose } v_1 = 1 = v_3 \\ &v_2 = -1 \end{aligned}$$

and the eigenspace $E_0 = \text{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

A is not diagonalizable.

Problem 7.3.21 - Find a 2×2 matrix A for which $E_1 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ and $E_2 = \text{span}\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\}$. How many such matrices are there?

Method 1: ~~B~~ Via diagonalization.

Strategy: Since $\lambda_1 = 1$, $\lambda_2 = 2$, A must be diagonalizable by Thm. 7.3.4. Hence there exists an S where $S^{-1}AS = D = \begin{bmatrix}1&0\\0&2\end{bmatrix}$. We construct S using the eigenvectors and calculate A.

Solution: For $D = \begin{bmatrix}1&0\\0&2\end{bmatrix}$, $S = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$ where $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\3\end{bmatrix}$. And since $S^{-1}AS = D$, we have $A = SDS^{-1}$. We get

$$A = SDS^{-1} = \underbrace{\begin{bmatrix}1&2\\2&3\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}}(-1)\begin{bmatrix}3&-2\\-2&1\end{bmatrix}$$

$$= \begin{bmatrix}1&4\\2&6\end{bmatrix}\begin{bmatrix}-3&2\\2&-1\end{bmatrix} = \begin{bmatrix}5&-2\\6&-2\end{bmatrix}.$$

Here, A is ~~unique~~ almost unique.

We can also choose, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, with $S = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 \end{bmatrix}$.

and $A = SDS^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}.$$

Hence A is in fact unique.

Method 2: Brute force.

Strategy: We solve $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for the 4 entries of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution: We get the partially uncoupled system of four unknowns in 4 equations:

$$\left.\begin{array}{l} A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{array}\right\} \Rightarrow \left.\begin{array}{c} a+2b=1 \\ c+2d=2 \\ 2a+3b=4 \\ 2c+3d=6 \end{array}\right\} \text{rearrange into 2 systems in 2 unknowns:}$$

$$\underbrace{\begin{array}{c} a+2b=1 \\ 2a+3b=4 \end{array}}_{b=-2,\ a=5} \qquad \underbrace{\begin{array}{c} c+2d=2 \\ 2c+3d=6 \end{array}}_{d=-2,\ c=6}$$

And $A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$ as before.

$\underline{A \text{ is unique}}$.

Problem 7.3.36 – Is the matrix $\begin{bmatrix} 0 & 4 \\ 5 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$?

Answer: No, their traces are different, and by Thm 7.3.5 (d) they must be the same for the matrices to be similar.

Problem 7.5.20 – Find the complex eigenvalues of $A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$.

Strategy: Solve the char. eqn over $\mathbb{C}$.

Solution: Here char eqn is:

$$\lambda^2 - tr(A)\lambda + \det(A) = \lambda^2 - 0\lambda + 1 = 0$$

which is solved by $\lambda = \frac{0 \pm \sqrt{0^2 - 4(1)(1)}}{2}$

or $\lambda = \pm i$

Problem 7.5.30 - (a) If $2i$ is an eigenvalue of a real 2×2 matrix A, find A^2.

Strategy: Use the fact that both eigenvalues are complex conjugates ($\lambda_1 = 2i$ $\lambda_2 = -2i$) to construct a general A. Then calculate A^2.

Solution: Since $\lambda = \pm 2i$, we know

$tr(A) = 2i + (-2i) = 0$, $det(A) = 2i(-2i) = -4$.

Since A has 0 trace, we know that for any element in the top left spot $a \in \mathbb{R}$, the bottom right is $-a$.

Write $A = \begin{bmatrix} a & s \\ r & -a \end{bmatrix}$ for $a \in \mathbb{R}$, $r \neq 0 \in \mathbb{R}$.

Then since $\det A = -a^2 - rs = -4$, we know $s = \frac{4-a^2}{r}$. So

$$A = \begin{bmatrix} a & \frac{4-a^2}{r} \\ r & -a \end{bmatrix}.$$

Then $A^2 = \begin{bmatrix} a & \frac{4-a^2}{r} \\ r & -a \end{bmatrix}\begin{bmatrix} a & \frac{4-a^2}{r} \\ r & a \end{bmatrix}$

$$= \begin{bmatrix} a^2 - \left(\frac{4-a^2}{r}\right) r & a\left(\frac{4-a^2}{r}\right) - \left(\frac{4-a^2}{r}\right)a \\ ar - ar & -r\left(\frac{4-a^2}{r}\right) - a^2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Hence no matter what the values of a, and $r \neq 0$ are, $A^2 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix}$, $\lambda = 2i$.

Problem 7.5.30 - (S) Give an example of a 2×2 matrix A such that all entries of A are non zero and $2i$ is an eigenvalue. Compute A^2 and check that your answer agrees with part a.

Solution: Choose $a \in \mathbb{R}$ anything non zero, and not ± 2, & $r \in \mathbb{R}$ anything: ~~$A = [$~~
choose $a = 1$, $r = 1$. Then $A = \begin{bmatrix} a & \frac{4-a^2}{r} \\ r & -a \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$