Name: ______________________________ **Section Number:** ____________

110.201 Linear Algebra
FALL 2013
MIDTERM EXAMINATION Solutions
October 11, 2013

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		30
2		30
3		15
4		25
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:** ____________

Question 1. [30 points] For the system $x - y + 3z = 1$, $y = -2x + 5$, $9z - x - 5y + 7 = 0$, do the following:

(a) Write the system in the matrix form $\mathbf{Ax} = \mathbf{b}$, for $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Strategy: Reorder the equations to line up the variables and then dig out the matrix $\mathbf{A}$.

Solution: First, note that the equations are not in an order to "see" the matrix equation. We reorder them to line up the variables as they occur:

$$\begin{array}{rrrrcr} \textbf{(I)} & x & -y & +3z & = & 1 \\ \textbf{(II)} & 2x & +y & & = & 5 \\ \textbf{(III)} & -x & -5y & +9z & = & -7 \end{array}$$

Then you can easily "pull out" the parts of the matrix equation:

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ -1 & -5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}.$$

(b) Write out the augmented matrix for this system and calculate its row-reduced echelon form.

Solution: The augmented matrix is just the 3×4 matrix $[\mathbf{A}|\mathbf{b}]$, or

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & 5 \\ -1 & -5 & 9 & -7 \end{array}\right].$$

Row reduction operations take us to

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & 5 \\ -1 & -5 & 9 & -7 \end{array}\right] \begin{array}{c} 2\textbf{(I)} - \textbf{(II)} \to \textbf{(II)} \\ \textbf{(I)} + \textbf{(III)} \to \textbf{(III)} \\ \Longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & 12 & -6 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & 12 & -6 \end{array}\right] \begin{array}{c} -\frac{1}{3}\textbf{(II)} \to \textbf{(II)} \\ 6\textbf{(II)} + \textbf{(III)} \to \textbf{(III)} \\ \Longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \textbf{(II)} + \textbf{(I)} \to \textbf{(I)} \\ \Longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This last aumented matrix is in reduced-row echelon form.

(c) Write out the complete set of solutions (if they exist) in vector form using parameters if needed.

Strategy: Use the resulting equations form the row-reduced echelon form of the augmented matrix to construct the solutions set. For each free-variable, associate a parameter and then write the other variable in terms of the free variable. Then create a vector version.

Solution: Using the row-reduced echelon form of the augmented matrix from the last part, we can write out the two non-trivial equations immediately: We get

$$x + z = 2$$
$$y - 2z = 1.$$

The variable z is a free-variable. Call it $z = t$. Then we get the three equations

$$x = 2 - t$$
$$y = 1 + 2t$$
$$z = t$$

in a parameterized form. In vector form, we get

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

(d) Calculate the inverse of the coefficient matrix $\mathbf{A}$ you found in part **(a)**, if it exists, or show that $\mathbf{A}^{-1}$ doesn't exist.

Strategy: Use parts **(a)** and **(b)** to show that the inverse cannot exist for $\mathbf{A}$.

Solution: From part **(a)**, we know

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ -1 & -5 & 9 \end{bmatrix}.$$

And from part **(b)**, we see its row-reduced echelon form is

$$\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Immediately, we know that $\text{rank}(\mathbf{A}_{3\times 3}) = 2$, so that $\mathbf{A}$ is not invertible. Hence $\mathbf{A}^{-1}$ does nto exist.

Question 2. [30 points] Let V be the subspace of $\mathbb{R}^4$ given by all solutions to the equation $2x_1 - x_2 + 3x_3 = 0$.

(a) What is the dimension of V?

Solution: There is little to do here. V is determined by a single linear equation in $\mathbb{R}^4$. Hence the set of solutions is a $4 - 1 = 3$-dimensional space. Hence $\dim(V) = 3$.

(b) Construct a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$, $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $V = \mathrm{im}(\mathbf{A})$. Then use $\mathbf{A}$ to construct a basis for $\mathrm{im}(\mathbf{A})$. You will need to verify that what you have is a basis.

Strategy: Find three sets of solutions to the equation for V in such a way that they are sufficiently different from each other by choosing various pairs to "cancel each other out", and render the other variables 0. Then use these solutions to form column vectors for the matrix $\mathbf{A}$. Since the column vectors always span the image of $\mathbf{A}$, is we can show the three columns of $\mathbf{A}$ are linearly independent, then they will form a basis of $\mathrm{im}(\mathbf{A})$.

Solution: We create one such set of solutions like in our strategy:

$$\begin{array}{rllll} \textbf{(i)} & x_1 = 1, & x_2 = 2, & x_3 = 0, & x_4 = 0 \\ \textbf{(ii)} & x_1 = 0, & x_2 = 3, & x_3 = 1, & x_4 = 0 \\ \textbf{(iii)} & x_1 = 0, & x_2 = 0, & x_3 = 0, & x_4 = 1 \end{array}.$$

We chose the last one since x_4 is not in the equation. It follows that any set of values for the four variables in which the first three are 0 and the last is not is automatically a solution.
We form the matrix $\mathbf{A}$ by writing each solution set above as a column and make these the columns of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, and the columsn of $\mathbf{A}$ span $\mathbf{A}$. To show that these columsn are linearly independent, we can do one of two things: (1) declare that they are since each has a non-zero column element not found in any of the other vectors (Tis is Theorem 3.2.5); Or (2), we can compute the row-reduced echelon form:

$$\mathrm{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here the rank of the matrix equals the number of columns. Thus the columsn are linearly independent.

(c) Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$, $T(\mathbf{x}) = \mathbf{B}\mathbf{x}$, where $V = \ker(\mathbf{B})$. Then use $\mathbf{B}$ to construct a basis for $\ker(\mathbf{B})$. You will need to verify that what you have is a basis.

Strategy: Like in class, we can simply use the dot product on vectors to construct the matrix $\mathbf{B}$. Once we have $\mathbf{B}$, we put it in its row-reduced echelon form, and construct elements of the kernel as in Theorem 3.3.8 from the text. And we can use the same criteria as in part **(b)** above to establish that we indeed have a basis.

Solution: We use the dot product to construct the equation in a matrix form:

$$T(\mathbf{x}) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{B}\mathbf{x}.$$

Note that for the matrix $\mathbf{B} = \begin{bmatrix} 2 & -1 & 3 & 0 \end{bmatrix}$, the redundant columns are the last three (why is this?). We use this and Theorem 3.3.8 to construct elements in teh kernel of $\mathbf{B}$: Express each redundant column as a linear combination of all preceding columns and use it to generate a vector. Note that these "columns" are really just 1-vectors.

Indeed, first we get $x_2 = -\frac{1}{2}x_1$. This generates the vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Then, we get $x_3 = x_1 - x_2$. This generates the vector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Lastly, x_4 cannot be written as a linear combination for the others (its coefficient is 0 in $\mathbf{B}$). But this immediately means that the vector $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \ker(\mathbf{B})$.

I leave it to you to check that these vectors are linearly independent. Follow part (b) for guidance.

Question 3. [15 points] For $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\mathbf{x}$ be a linear transformation. Do the following:

(a) Write this transformation as a composition of a scaling and an orthogonal projection.

Strategy: Use the form for an orthogonal projection to see whether there is a constant we can pull out to render T an orthogonal projection. If so, then this constant is the scaling, which we write a s a matrix.

Solution: The form for an orthogonal projection is given by

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(\mathbf{x}) = \mathbf{A}\mathbf{x}, \text{ where } \mathbf{A} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix},$$

for some unit vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ in the image of T. The fact that $\mathbf{u} \in \text{im}(T)$ means that $u_1^2 + u_2^2 = 1$
In our case, the matrix is in the right form for $u_1 = 2$ and $u_2 = -1$, but then $u_1^2 + u_2^2 = 4 + 1 = 5 \neq 1$. Hence T is NOT an orthogonal projection. However, we can write

$$T(\mathbf{x}) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\mathbf{x} = 5\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\mathbf{x}.$$

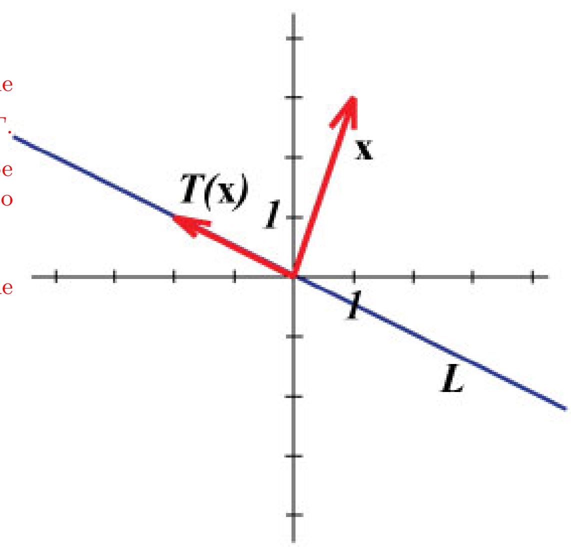
Then, the right-hand-side of above is a constant multiple of a new matrix corresponding to a linear transformation in the orthogonal projection format with $u_1 = \frac{2}{\sqrt{5}}$ and $u_2 = -\frac{1}{\sqrt{5}}$. Then $u_1^2 + u_2^2 = \frac{4}{5} + \frac{1}{5} = 1$. And since we can write any constant multiple as a scaling, we get

$$T(\mathbf{x}) = 5\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\mathbf{x} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\mathbf{x}.$$

(b) Find the equation of the line $L = \text{im}(T)$ and carefully draw L, $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ on the graph provided.

Solution: Above, we established that the vector $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in the image of T. Hence we can immediately calculate the slope of the line L containing $\mathbf{u}$, as $\frac{\Delta y}{\Delta x} = -\frac{1}{2}$ so that the equation of the line L is $y = -\frac{1}{2}x$.

The vectors and L are graphed, with the line in blue and the two vectors in red.



Question 4. [25 points] Suppose we know for a linear transformation T of $\mathbb{R}^2$ that $T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$ and $T\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$. Do the following:

(a) Find the matrix $\mathbf{A}$ so that $T(\mathbf{x})=\mathbf{A}\mathbf{x}$.

Strategy: The columns of the matrix $\mathbf{A}$ are just the transformation's effect on the standard vectors. The vectors given are linear combinations of standard vectors. We use this to solve for the columns of $\mathbf{A}$.

Solution: The matrix $\mathbf{A}$, relative to the standard basis, is given as $\mathbf{A}=\begin{bmatrix}| & |\\ T(\mathbf{e}_1) & T(\mathbf{e}_2)\\ | & |\end{bmatrix}$. So

$$T\begin{bmatrix}1\\1\end{bmatrix}=T(\mathbf{e}_1+\mathbf{e}_1)=T(\mathbf{e}_1)+T(\mathbf{e}_2)=\begin{bmatrix}3\\5\end{bmatrix}$$
$$T\begin{bmatrix}-1\\2\end{bmatrix}=T(-\mathbf{e}_1+2\mathbf{e}_1)=-T(\mathbf{e}_1)+2T(\mathbf{e}_2)=\begin{bmatrix}0\\1\end{bmatrix}.$$

We can solve this as a 2×2 system: Add the two last equations together to get $3T(\mathbf{e}_2)=\begin{bmatrix}3\\6\end{bmatrix}$, so that $T(\mathbf{e}_2)=\begin{bmatrix}1\\2\end{bmatrix}$. PLug this back into the first equation to get $T(\mathbf{e}_1)=\begin{bmatrix}2\\3\end{bmatrix}$. Then $\mathbf{A}=\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$.

(b) Given the basis $\mathcal{B}=\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix}\right\}$, find the matrix $\mathbf{B}$ so that $T\,[\mathbf{x}]_{\mathcal{B}}=B\,[\mathbf{x}]_{\mathcal{B}}$.

Strategy: The short way to do this is to compute the change-of-basis matrix $\mathbf{S}$ using $\mathcal{B}$. Then $\mathbf{AS}=\mathbf{SB}$, or $\mathbf{B}=\mathbf{S}^{-1}\mathbf{AS}$.

Solution: Given $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$, we can write

$$\mathbf{S}=\begin{bmatrix}| & |\\ \mathbf{v}_1 & \mathbf{v}_2\\ | & |\end{bmatrix}=\begin{bmatrix}1 & 3\\-2 & 3\end{bmatrix}.$$

Then $\mathbf{S}^{-1}=\frac{1}{9}\begin{bmatrix}3 & -3\\2 & 1\end{bmatrix}$, so that

$$\begin{aligned}\mathbf{B}=\mathbf{S}^{-1}\mathbf{AS}&=\frac{1}{9}\begin{bmatrix}3 & -3\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\-2 & 3\end{bmatrix}\\&=\frac{1}{9}\begin{bmatrix}3 & -3\\2 & 1\end{bmatrix}\begin{bmatrix}0 & 9\\-1 & 15\end{bmatrix}=\frac{1}{9}\begin{bmatrix}3 & -18\\-1 & 33\end{bmatrix}=\begin{bmatrix}\frac{1}{3} & -2\\-\frac{1}{9} & \frac{11}{3}\end{bmatrix}.\end{aligned}$$

(c) Find the $\mathcal{B}$-coordinates of the vector $\mathbf{x}=\begin{bmatrix}2\\5\end{bmatrix}$.

Solution: Since we know that $\mathbf{S}\,[\mathbf{x}]_{\mathcal{B}}=\mathbf{x}$, and we know that $\mathbf{S}$ is invertible, we simply compute, using part (b) above

$$[\mathbf{x}]_{\mathcal{B}}=\mathbf{S}^{-1}\mathbf{x}=\frac{1}{9}\begin{bmatrix}3 & -3\\2 & 1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}=\frac{1}{9}\begin{bmatrix}-9\\9\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}.$$

Of course, with a little inspection, you can also simply see that

$$\begin{bmatrix}2\\5\end{bmatrix}=-1\cdot\begin{bmatrix}1\\-2\end{bmatrix}+1\cdot\begin{bmatrix}3\\3\end{bmatrix}.$$

Hence the coordinates are $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}-1\\1\end{bmatrix}$.