Class 12: 9/30/13

VIII

Thm (3.2.8) Given $A_{n\times m}$, the column vectors are linearly independent iff $\ker(A) = \{\vec{0}\}$.

Note: This relates the image of A (span of the column vectors) to the kernel of A even though they "live" in different places!

pf. ($\Leftarrow$) <u>if $\ker(A)=\{\vec{0}\}$, then cols. are lin. indep.</u>

If $\ker(A)=\{\vec{0}\}$, then the only solution to $A\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$. But

$$(*) \quad A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underbrace{x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}}.$$

Since $\ker(A)=\{\vec{0}\}$, this relation is trivial.

Hence by Thm 3.2.7, col. vectors are lin. indep.

($\Rightarrow$) <u>if cols are lin. indep. then $\ker(A)=\{\vec{0}\}$.</u>

If cols are lin. indep., then (*) is a trivial relation. Then cols of A form a basis for im(A) (they already span A). Then $\text{rank}(A)=m$. Then $m\le n$, for $A_{n\times m}$.

By Thm 3.1.7, then, $\ker(A)=\{\vec{0}\}$. ∎

Finding a basis for each of ker(A) and im(A), and determining dimension?

Key facts

(I) Given any A, the solutions to ~~ker~~ $A\vec{x}=\vec{0}$ are exactly the same as that of $B\vec{x}=\vec{0}$, where $B=\text{rref}(A)$. (why?)

That is, $\ker(A)=\ker(B)=\ker(\text{rref}(A))$

- Once in rref, give every free variable a parameter and write all variables in terms of free variables.
- ~~Restate~~ Rewrite these equations as vector equations and pull out the parameters.
- The constant vectors of this vector eqn will form a basis of $\ker(\text{rref}(A))=\ker(A)$.
- The number of these vectors is the dimension of $\ker(A)$.

See example 1a, page 136-7.

ex. Back to example 3 of previous lecture

Here image of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is $\text{span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right)$.

and since $\text{im}(A) \subset \mathbb{R}^2$, it should take no more than 2 vectors to span im A.

Q: Does $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ span (im A)?

A: It does, if ① $\forall k \in \mathbb{R}$ $\begin{bmatrix} 1 \\ 4 \end{bmatrix} \neq k\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. True.

and ② We can write $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Here

$$\underbrace{k_1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Q: Is there a solution to system?

A: $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & 6 \end{array}\right] \to \begin{array}{l} k_1 = -1 \\ k_2 = 2 \end{array}$

Hence yes, and $\text{span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \text{im}\, A = \mathbb{R}^2$.

$\blacksquare$

exercise: Show $\left\{\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$ are also bases for im(A).

Some facts about bases

Thm 3.3.2

① All bases of a linear subspace $V \subset \mathbb{R}^n$ have the same number of vectors.

② The number of vectors in a basis is called the dimension of $V \subset \mathbb{R}^n$, denoted $\dim(V)$.

③ If $\dim(V) = m$, where $V \subset \mathbb{R}^n$ is a subspace, then

ⓐ we can find at most m lin. indep. vectors.

ⓑ we need at least m vectors to span V.

ⓒ If m vectors ~~span~~ in V ~~and~~ are linearly indep. then their span is V, and they form a basis.

Now we know that dimension is a well defined concept, and corresponds to the size of a subspace by the # of lin indep. vectors in span.

(II) A basis for image(A) is just the column vectors which are not redundant.

Key fact: The image(A) $\neq$ image(rref(A)).

But (1) it is of the same size, and

(2) The redundant columns of rref(A) are the same ones as those of A!

Method:
- Put A in rref.
- Note position of every ~~[illegible]~~ column vector with a leading 1 in it.
- Basis for im(A) is this set of vectors.

(All vectors without a leading 1 in them are redundant)

See example 16, page 136-7.

Special Note: One can see immediately that

(a) $\dim(\ker A) = \#$ of free variables in rref(A)

(b) $\dim(\operatorname{Im} A) = \#$ of cols. w/ leading 1 in rref(A).

(c) $\dim(\operatorname{Im}(A)) + \dim(\ker(A)) = m$ for $A_{n\times m}$.