Class 11: 9/27/13

ex. Example 3.2.3 pg 124.

Consider the plane V defined by the eqn $x_1 + 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$.

(a) Find a matrix A, where $V = \ker(A)$

(b) Find a matrix B, where $V = \operatorname{Im}(B)$

Strategy: Construct the linear transformation T_A and T_B.

Solution (a) Consider the transformation

$T_A : \mathbb{R}^3 \to \mathbb{R}$ given by $T_A(\vec{x}) = \vec{x} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Writing out the dot product,

we have $T(\vec{x}) = x_1 + 2x_2 + 3x_3$

and since $0 \in \mathbb{R}$, the equation $T(\vec{x}) = 0$

defines $\ker(T)$, precisely the equation of V.

To find the matrix, rearrange the dot product

$T_A(\vec{x}) = [1 \quad 2 \quad 3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then $A = [1 \quad 2 \quad 3]$

and $\ker(A) = V$.

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(b) if we can find 2 vectors $\vec{v}_1, \vec{v}_2$ both in V (so they satisfy the eqn) which are not multiples of each other, then

$\text{span}(\vec{v}_1, \vec{v}_2) \subset V$.

We will find in the next section that here $\text{span}(\vec{v}_1, \vec{v}_2)$ will be a 2-dim. subspace of V ~~[illegible]~~. Hence

$$\text{span}(v_1, v_2) = V \subset \mathbb{R}^3.$$

The book "chooses" $x_1 = -2,\ x_2 = 1,\ x_3 = 0$

and $x_1 = -3,\ x_2 = 0,\ x_3 = 1$

as 2 solutions to $x_1 + 2x_2 + 3x_3 = 0$.

Let $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Here $\nexists\, k \in \mathbb{R}$, where $\vec{v}_1 = k\vec{v}_2$. Hence $\text{span}(\vec{v}_1, \vec{v}_2) = V$

Create $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(\vec{x}) = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then $\text{im}(T) = \text{im}(B) = \text{span of columns of } \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

But then $\text{im}(B) = V$. ∎

ex. 3.2.4

Find the minimum number of vectors that span the image of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

Strategy: Look for ways that some columns can be written as linear combinations of others. Eliminate all of these to find image as the span of the rest.

Solution

Image$(A) = $ span$(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}$$

But $\vec{v}_2 = 2\vec{v}_1$, and $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$

Hence $\text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_3)$

And since $\vec{v}_1 \notin \text{span}(\vec{v}_3)$ $(\vec{v}_1 \neq k\vec{v}_3 \ \forall k \in \mathbb{R})$

we have $\boxed{\text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_3)}$ ∎

Note: Look at $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, particularly where the leading terms are. Can you form a conjecture?

<u>Defs</u>

(A) In a list of vectors $\vec{v}_1, \dots, \vec{v}_m$, any $\vec{v}_i$ that can be written as a linear combination of the others is called <u>redundant</u>.

ex. In previous example, $\vec{v}_2$ and $\vec{v}_4$ are redundant in $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

(B) The vectors $\vec{v}_1, \dots, \vec{v}_m$ are called <u>linearly independent</u> if none are redundant, and <u>linearly dependent</u> if at least one is.

(C) A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ in a subspace $V \subset \mathbb{R}^n$ are called a <u>basis</u> for V if $\text{span}(\vec{v}_1, \dots, \vec{v}_m) = V$ <u>and</u> are linear independent.

ex. In previous example, $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent, and $\vec{v}_1, \vec{v}_3$ form a basis for the subspace $\text{im}(A) \subset \mathbb{R}^3$.

Note: For $\vec{v}_1, \ldots, \vec{v}_m$, the equation

$$c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = 0$$

is called a <u>linear relation</u> among vectors.

(a) Called <u>trivial</u> if only solution is $c_1 = \cdots = c_m = 0$ (it is always a solution).

(b) If there are other solutions (where some or all c_i's are nonzero), relation is <u>nontrivial</u>.

<u>Thm</u> 3.2.7 The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent iff there is a nontrivial relation between them.

<u>Thm</u> 3.2.8 Given $A_{n \times m}$, the column vectors are linearly independent iff $\ker(A) = \{\vec{0}\}$.

Note: This relates the image of A (span of column vectors) to the kernel of A.