Class 10: 9/25/13

ex. How to find $\ker(A)$ and $\text{Im}(A)$, for $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$.

(A) For $\ker(A)$, simply solve $A\vec{x} = \vec{0}$.

<u>ex1</u>: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Here $\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

Find $\ker(A)$.

Strategy: Calculate $\text{rref}(A)$ and use it to find a way to parameterize all solutions to $A\vec{x} = \vec{0}$.

<u>Solution</u>: $A\vec{x} = \vec{0}$ corr to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right].$$

The system with A replaced by $\text{rref}(A)$ has the same solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(A) cont'd.

ex1: cont'd.

solution: cont'd.

This new system reduces to the equation

$$x_1 + 2x_2 = 0 \quad \text{or} \quad x_1 = -2x_2$$

Here, x_2 plays the role of a free variable, and if we assign it a parameter value: $x_2 = t \in \mathbb{R}$.

Then $x_1 = -2x_2 = -2(t) = -2t$.

Any vector of the form

$$\vec{x} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

will solve $A\vec{x} = \vec{0}$ (check this!)

Hence $\ker(A) = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. ∎

(A) cont'd.

ex2: Find $\ker\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right)$.

Strategy: Use the same procedure as that used in example 1 above, noting that by the previous lecture, since A is a 2×3 matrix with more columns than rows, it must be the case that $\ker(A)$ will contain an infinite number of solutions.

Solution: Convert $A\vec{x}=\vec{0}$ to its equivalent rref form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

with equations
$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$

Parameterize the free variable $x_3 = t$.

(A) cont'd.

ex2: cont'd.

Solution: cont'd.

We get $x_3 = t$, $x_1 = t$, $x_2 = -2t$.

Then any $\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ satisfies $A\vec{x} = \vec{0}$

and $\ker(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$. $\blacksquare$

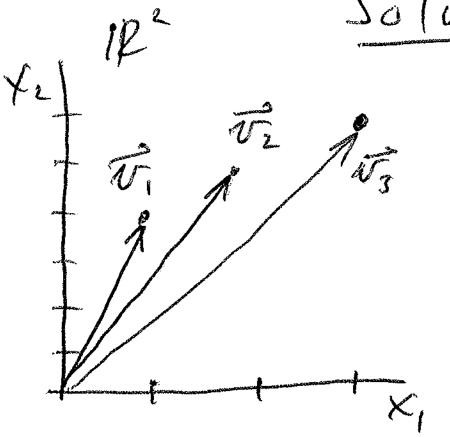
(B) The image of A is just the span of its columns.

ex3: Describe the image of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ under $T: \mathbb{R}^3 \to \mathbb{R}^2$; $T(\vec{x}) = A\vec{x}$.

Strategy: We describe $\text{im}(A)$ as a subspace of $\mathbb{R}^2$ by looking at the column vectors of A.

(B) cont'd.

<u>ex 3</u> cont'd.



<u>Solution</u>: Here $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

$$= x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3$$

We will see in the near future that if 2 vectors in $\mathbb{R}^2$ point in different directions, then their span is all of $\mathbb{R}^2$.

For now, note that if we choose only $\vec{v}_1$ and $\vec{v}_2$, the new system

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

will have a unique solution for any choice of $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$. ($\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is invertible)

This means that $\text{span}(\vec{v}_1, \vec{v}_2) = \mathbb{R}^2$, so that $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^2$. Thus $\text{Image}(A) = \mathbb{R}^2$ ($T(\vec{x}) = A\vec{x}$ is onto.)

New Def

A subset W of the vector space $\mathbb{R}^n$ is called a (linear) subspace of $\mathbb{R}^n$ if it has the following properties:

ⓐ $\vec{0} \in W \subset \mathbb{R}^n$.

ⓑ W is closed under vector addition in $\mathbb{R}^n$
(if $\vec{w}_1, \vec{w}_2 \in W$, then so is $\vec{w}_1 + \vec{w}_2$)

ⓒ W is closed under scaler multiplication in $\mathbb{R}^n$
(if $\vec{w} \in W$, then so is $k\vec{w}$ for all $k \in \mathbb{R}$.)

Notes: $W \subset \mathbb{R}^n$ is a subspace if
whenever $\vec{w}_1, \vec{w}_2 \in W \subset \mathbb{R}^n$, then so is $\text{span}(\vec{w}_1, \vec{w}_2)$.

Thm 3.2.2 For $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A_{n \times m}\vec{x}$ a linear transformation, we have

① $\ker(T) = \ker(A)$ is a subspace of $\mathbb{R}^m$

② $\text{im}(T) = \text{im}(A)$ is a subspace of $\mathbb{R}^n$.

ex. Q1: Is the x-axis a subspace of $\mathbb{R}^2$?

A1: We can describe the x-axis as

$$\text{x-axis} = \left\{ \vec{v} \in \mathbb{R}^2 \;\middle|\; \vec{v} = k\begin{bmatrix}1\\0\end{bmatrix} \text{ for some } k \in \mathbb{R} \right\}$$

check the 3 properties .... Yes, it is.

ex. Q2: Is the line $y = x - 1$ a linear subspace of $\mathbb{R}^2$?

A2: No, $\vec{0}$ is not on the graph.

(Here, $x=0, y=0$ is NOT a solution to eqn).

We can list all possible linear subspaces of $\mathbb{R}^2$.

(I) $\{\vec{0}\}$ called the 0-dim. subspace. (How to write it?)

(II) $\left\{\begin{matrix}\text{any line}\\ \text{through}\\ \text{origin}\end{matrix}\right\}$ called 1-dim subspaces. (Form: $ax + by = 0$ why?)

(III) $\mathbb{R}^2$ a 2-dim subspace.

How about in $\mathbb{R}^3$?