Class 9 : 9/23/13

Chapter 3 Subspaces of $\mathbb{R}^n$

Section 3.1 The Image of a linear transf.

<u>Def.</u> Given $f: X \to Y$ a function on sets, the set $f(X) \overset{\subset Y}{\text{is}}$ called the image of f, and

$$f(X) = \text{image of } f = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}$$

ex. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2\sin x$. Then
$f(\mathbb{R}) \subset \mathbb{R}$, $\quad f(\mathbb{R}) = [-2, 2] \subset \mathbb{R}$.

ex. Let $T_L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of $\mathbb{R}^2$ which takes $\mathbb{R}^2$ to ~~the~~ $L = \{x\text{-axis}\}$.

Here $T(\vec{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ for $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ a unit vector on x-axis. Choose $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\Longrightarrow T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}. \quad \text{Here } f(\mathbb{R}^2) = L \subset \mathbb{R}^2$$

Q: Redo this with $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$. Here T is invertible. Hence it is 1-1 and <u>onto</u>.

Being onto, we have $T(\mathbb{R}^2) = \mathbb{R}^2$ (all of Y).

ex. Let $g: \mathbb{R} \to \mathbb{R}^2$, $g(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

Q1: Is g a linear transformation? (does $g(t_1+t_2) = g(t_1) + g(t_2)$?

Q2: Find $g(\mathbb{R})$.

Q3: Is g invertible?

<u>Definition</u> Given $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$. The set of all linear combinations $c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ is called the <u>span</u> of the vectors. That is,

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \left\{ \vec{w} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{w} \text{ for some} \\ \text{choice of } c_1, \dots, c_m \in \mathbb{R}. \end{array} \right\}.$$

<u>Thm</u> The image of a linear transformation, $T(\mathbb{R}^m)$, for $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A_{n\times m}\vec{x}$ is the span of the column vectors of A. ~~Hence~~ Denoted $\text{im}(T)$ or $\text{Im}(A)$.

~~$T(\mathbb{R}^m) =$~~

Hence $T(\mathbb{R}^m) = \text{im}(T) = \text{im}(A)$. This is because

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$$

looks exactly like def above.

Properties of the Image of a lin. transf.

$T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A_{n\times m}\vec{x}$.

① $\vec{0} \in \mathbb{R}^n$ is ALWAYS in $\mathrm{im}(T)$.

② The image is closed under addition: if $\vec{v}_1 \in \mathrm{im}(T)$ and $\vec{v}_2 \in \mathrm{Im}(T)$, then $\vec{v}_1 + \vec{v}_2 \in \mathrm{im}(T)$.

③ The image is closed under scalar multiplication: if $\vec{v}_1 \in \mathrm{Im}(T)$, then $k\vec{v}_1 \in \mathrm{im}(T)$ $\forall k \in \mathbb{R}$.

Note: ② and ③ imply ①: if $\vec{v}_1 \in \mathrm{Im}(T)$, then so is $(-1)\vec{v}_1$, ~~and so~~ by ③ and by ② so is $\vec{v}_1 + (-1)\vec{v}_1 = \vec{0} \in \mathrm{Im}(T)$.

Note: By ② and ③, all linear combination of vectors in $\mathrm{im}(T)$ are in $\mathrm{im}(T)$.

Note: Let $\vec{v}_1, \vec{v}_2 \in \mathrm{im}(T)$, for $T: \mathbb{R}^m \to \mathbb{R}^n$ linear. Then $\mathrm{span}(\vec{v}_1, \vec{v}_2) \subset \mathrm{im}(T)$.

<u>Def</u> The <u>kernel</u> of a linear transf. $T : \mathbb{R}^m \to \mathbb{R}^n$
$T(\vec{x}) = A_{n\times m}\vec{x}$, denoted $\ker(T)$ or $\ker(A)$
is the set of all solutions $\vec{x} \in \mathbb{R}^m$ of
the equation $T(\vec{x}) = \vec{0}$, or $A\vec{x} = \vec{0}$.

<u>Notes</u> ① The kernel of T is also called the
<u>null space</u> of ~~A~~ A.

② The image of T is also called the
<u>column space</u> of A.

③ $\mathrm{im}(T)$ is ALWAYS a subset of the
receiving space $\mathbb{R}^n$, for $T : \mathbb{R}^m \to \mathbb{R}^n$

④ $\ker(T)$ is ALWAYS a subset of the
sending space (domain) $\mathbb{R}^m$.

That is, for $T : \mathbb{R}^m \to \mathbb{R}^n$ linear,
$\mathrm{im}(T) \subset \mathbb{R}^n$, $\ker(T) \subset \mathbb{R}^m$.

properties of the kernel

For a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$

(a) $\vec{0} \in \ker(T) \subset \mathbb{R}^m$.

(b) The kernel is closed under addition:
if $\vec{v}, \vec{w} \in \ker(T)$, then $\vec{v} + \vec{w} \in \ker(T)$.

(c) kernel is closed under scalar multiplication
if $\vec{v} \in \ker(T)$, then so is $k\vec{v}$, $\forall k \in \mathbb{R}$.

pf. (a) $A\vec{x} = \vec{0}$ always has the solution $\vec{x} = \vec{0} \in \mathbb{R}^m$.

(b) The transformation is linear, so if $\vec{v}, \vec{w} \in \ker(T)$

$$A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{0} + \vec{0} = \vec{0}$$

$\vec{w} \in \ker(T)$ so $A\vec{w} = \vec{0}$.

$\vec{v}$ solves $A\vec{v} = \vec{0}$ already since $\vec{v} \in \ker(T)$.

(c) $$A(k\vec{v}) = \underbrace{(Ak)\vec{v}}_{\text{matrix mult is assoc}} = k(A)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$$

$\vec{v} \in \ker(A)$, so $A\vec{v} = \vec{0}$

Note: Whenever 2 vectors $\vec{v}, \vec{w} \in \ker(T)$, then <u>all</u> linear combinations of $\vec{v}, \vec{w}$, or $k_1\vec{v} + k_2\vec{w}$ are also in $\ker(T)$. When $\vec{v}, \vec{w} \in \ker(T)$, then $\text{span}(\vec{v}, \vec{w}) \subset \ker(T)$.

Some final notes

(I) Sometimes $\ker(A) = \{\vec{0}\}$, (the zero-vector is the only vector in the kernel.)

(a) For $A_{n \times m}$, $\ker(A) = \{\vec{0}\}$ iff $\operatorname{rank}(A) = m$.

(b) if $\ker(A) = \{\vec{0}\}$, then it must be the case that $m \le n$ (why?) Think $\operatorname{rref}(A)$.

(c) if $m > n$ (more columns than rows, or more variables than equations), then there always exist more vectors in $\ker(A)$.

(d) if $m = n$, then $\ker(A) = \{\vec{0}\}$, iff A is invertible.

(II) Thm The following statements are equivalent for $A_{n \times n}$

i) A is invertible

ii) $A\vec{x} = \vec{b}$ has a unique soln for all $\vec{b} \in \mathbb{R}^n$

iii) $\operatorname{rref}(A) = I_n$

iv) $\operatorname{rank}(A) = n$

v) $\operatorname{im}(A) = \mathbb{R}^n$

vi) $\ker(A) = \{\vec{0}\}$.