Class 8: 9/20/13

From last class, we ended with

Thm 2.4.3 A square matrix $A_{n\times n}$ is invertible iff $rref(A) = I_n$

Today we start with a few more facts about invertible matrices.

ex. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Is A invertible?

Its $rref(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Hence no by Thm 2.4.3.

But $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible since $rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Do these calculations!!!

Thm 2.4.4 Let $A_{n\times n}$ be a matrix

(a) If A is noninvertible, then for a choice of $\vec{b}$, either $A\vec{x}=\vec{b}$ has no solutions or an infinite number.

Q: What is the contrapositive of this statement?

(b) Let $\vec{b}=\vec{0}$. The system $A\vec{x}=\vec{0}$ ALWAYS has a solution (what is it?)

If A is invertible, it is the ONLY solution

If A is not invertible, there is an infinite number of them.

Combine these to get one statement:

A is invertible iff $A\vec{x}=\vec{0}$ has a unique solution.

Q: How to find A^{-1} for an invertible A?

A: Really it is just a matter of taking the system $A\vec{x}=\vec{y}$ and solving for $\vec{x}$ as a function of $\vec{y}$. Then the coefficients of $\vec{y}$ are A^{-1}.

That is, if you can take the system

$$(*) \qquad A\vec{x}=\vec{y}, \text{ or } \qquad \begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \qquad \ddots \qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = y_n \end{matrix}$$

and slowly turn it into the system

$$(\star) \qquad \vec{x} = B\vec{y}, \text{ or } \qquad \begin{matrix} x_1 = b_{11}y_1 + \cdots + b_{1n}y_n \\ \vdots \qquad \ddots \qquad \vdots \\ x_n = b_{n1}y_1 + \cdots + b_{nn}y_n \end{matrix}$$

Then you have "found" a way of going back from $\vec{y}$ to $\vec{x}$ via a linear transfo.

- This is what the inverse of a lin transf does.
- Here $B = A^{-1}$

ex. Given $A\vec{x}=\vec{y}$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, find A^{-1}.

<u>Strategy</u>: We turn the system $(*)$ into the system $(\star)$, and note that $B = A^{-1}$!

ex. cont'd.

Here $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff \left| \begin{matrix} 2x_1 + x_2 = y_1 \\ x_1 + x_2 = y_2 \end{matrix} \right| \begin{matrix} (\mathrm{I}) \\ (\mathrm{II}) \end{matrix}$

This is (*) above for this problem. We rewrite to (A) using back notation from Section 1.1:

$$\left| \begin{matrix} 2x_1 + x_2 = y_1 \\ x_1 + x_2 = y_2 \end{matrix} \right| \begin{matrix} (\mathrm{I}) - (\mathrm{II}) \\ \end{matrix} \longrightarrow \left| \begin{matrix} x_1 = y_1 - y_2 \\ x_1 + x_2 = y_2 \end{matrix} \right| \begin{matrix} \\ (\mathrm{II}) - (\mathrm{I}) \end{matrix}$$

$$\longrightarrow \left| \begin{matrix} x_1 = y_1 - y_2 \\ x_2 = y_2 - (y_1 - y_2) \end{matrix} \right| = \left| \begin{matrix} x_1 = y_1 - y_2 \\ x_2 = -y_1 + 2y_2 \end{matrix} \right|$$

which corresponds to (A) and the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{or} \quad \vec{x} = B\vec{y}.$$

Here $B = A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

In general, one can find A^{-1} from an invertible $A_{n\times n}$ via the following:

1. Create a new $n\times 2n$ matrix $[A_{n\times n} \mid I_n]$
2. Compute $\mathrm{rref}[A \mid I_n]$.

At the end, you will get a matrix of the form $[I_n \,\vdots\, B_{n\times n}]$.

Then $B = A^{-1}$, and is the coefficient matrix of (A).

Some additional Matrix inverse facts

- Thm 2.4.6 For $A_{n\times n}$ invertible, $AA^{-1} = A^{-1}A = I_n$. (see previous example from last lecture).
- Thm 2.4.7 For $A_{n\times n}, B_{n\times n}$ invertible, we have $(AB)^{-1} = B^{-1}A^{-1}$ (order matters!). why?
- Thm 2.4.9 For $A_{2\times 2}$ ~~invertible~~, $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

  (a) A is invertible iff $ad-bc \neq 0$. (The quantity $ad-bc = \det A$ = determinant of A.)

  (b) If invertible, then

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

exercise: Show this by the method above!

Chapter 3 Subspaces of $\mathbb{R}^n$

Section 3.1 The Image of a linear transf.

<u>Def.</u> Given $f: X \to Y$ a function on sets, the set $f(X) \subset Y$ is called the image of f, and

$$f(X) = \text{image of } f = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}$$

ex. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2\sin x$. Then
$f(\mathbb{R}) \subset \mathbb{R}$, $f(\mathbb{R}) = [-2, 2] \subset \mathbb{R}$.

ex. Let $T_L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of $\mathbb{R}^2$ which takes $\mathbb{R}^2$ to ~~the~~ $L = \{x\text{-axis}\}$.
Here $T(\vec{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ for $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ a unit vector on x-axis. Choose $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\Longrightarrow T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}. \quad \text{Here } f(\mathbb{R}^2) = \cancel{\ } L \subset \mathbb{R}^2$$

Q: Redo this with $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$. Here T is invertible. Hence it is 1-1 and <u>onto</u>.
Being onto, we have $T(\mathbb{R}^2) = \mathbb{R}^2$ (all of Y).