Class 7: 9/18/13

Let X, Y be 2 arbitrary sets, and $T: X \to Y$ a function.

<u>Def</u> T is called <u>invertible</u> if the equation $T(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. When this is the case, the function $T^{-1}: Y \to X$, $T^{-1}(y) = x$ is called the <u>inverse</u> of T.

<u>Notes</u> ① The <u>fact</u> that $T(x) = y$ has a unique solution $x \in X$ for each choice of $y \in Y$ expresses T as both 1-1 and onto Y:

ⓐ That given some choice $y \in Y$, that $T(x) = y$ has a <u>unique</u> solution is <u>1-1</u>.

ⓑ That there is a solution to $T(x) = y$ <u>for every</u> $y \in Y$ is <u>onto</u>.

② Notice that $(T^{-1} \circ T)(x) = T^{-1}(T(x)) = T^{-1}(y) = x$

and $(T \circ T^{-1})(y) = T(T^{-1}(y)) = T(x) = y$

Notes (cont'd).

③ If T is invertible, then ANY function $f: \underline{Y} \to \underline{X}$, where $(f \circ T)(x) = x$ and $(T \circ f)(y) = y$, is called an inverse to T, and $f = T^{-1}$.

④ If T is invertible, then so is T^{-1}, and $(T^{-1})^{-1} = T$.

⑤ There is no mention or idea here as to how to find such a T^{-1} when T is invertible. Only that it exists.

⑥ If $\underline{X} = \underline{Y}$ (as sets), then $T: \underline{X} \to \underline{X}$, and if T is invertible, then we typically use the independent variable x for both ~~the~~ T and T^{-1}.

ex. (Pre-calc) Given $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 2$, we write $y = 3x - 2$, and solve for x as a function of y to create the inverse. $\frac{y+2}{3} = x$. Then, we switch the role of x, y to get $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$.

ex. (cont'd)

Here then $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x-2)$

$$= \tfrac{1}{3}(3x-2) + \tfrac{2}{3} = x$$

and $(f \circ f^{-1})(x) = f(f^{-1}(x)) = 3\left(\tfrac{1}{3}x + \tfrac{2}{3}\right) - 2 = x$

By the above definition, a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A_{n \times m}\vec{x}$ is <u>invertible</u> if $A\vec{x} = \vec{y}$ has a unique solution for every $\vec{y} \in \mathbb{R}^n$.

(Back to systems of eqns and the types of solutions they have).

<u>ex</u>. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$ is <u>invertible</u> and $T^{-1}(\vec{x}) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\vec{x}$ its inverse.

Here $(T^{-1} \circ T)(\vec{x}) = T^{-1}(T\vec{x}) = T^{-1}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$$= T^{-1}\left(\begin{bmatrix} 2x+y \\ x+y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2x+y \\ x+y \end{bmatrix}$$

$$= \begin{bmatrix} (2x+y) - (x+y) \\ -(2x+y) + 2(x+y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}.$$

ex (cont'd.)

Also show $(T \circ T^{-1})(\vec{x}) = \vec{x}$.

Notice also the following:

$$(T^{-1} \circ T)(\vec{x}) = T^{-1}(T(\vec{x})) = T^{-1}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$\overset{\text{assoc.}}{=} \left(\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= \begin{bmatrix} 1(2) - 1(1) & 1(1) - 1(1) \\ -1(2) + 2(1) & -1(1) + 2(1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2.}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}.$$

Q: How do we know whether a linear transf. has an inverse without actually solving for it?

Def. A square matrix A is said to be <u>invertible</u> if the linear transf. $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\vec{x}) = A\vec{x}$ is invertible. Then T_A^{-1} exists and we write the matrix for T^{-1} as A^{-1}.

For $T(\vec{x}) = A\vec{x} = \vec{y}$, we have $T^{-1}(\vec{y}) = A^{-1}\vec{y} = \vec{x}$.

Question to think about

Can a matrix be invertible if it is not square?

Hint: ① Can $T:\mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A_{n\times m}\vec{x}$ be 1-1 if $m>n$?

② Can T be onto if $m<n$?

Thm 2.4.3 A square matrix $A_{n\times n}$ is invertible iff $\mathrm{rref}(A) = I_n$.

In this case, then, for any choice of $\vec{y}\in\mathbb{R}^n$, we woud write the augmentation matrix for $A\vec{x}=\vec{b}$ $[A \mid \vec{b}]$. Then ~~reduced~~ row reduction would look like $[I_n \mid \vec{x}]$ where $\vec{x}$ is the solution to $A\vec{x}=\vec{b}$. Think about this.

Thm 2.4.4 Let $A_{n \times n}$ be a matrix

(a) If A is noninvertible, then for a choice of $\vec{b}$, either $A\vec{x}=\vec{b}$ has no solutions or an infinite number.

(b) Let $\vec{b}=\vec{0}$. The system $A\vec{x}=\vec{0}$ ALWAYS has a solution (what is it?)

If A is invertible, it is the ONLY solution

If A is not invertible, there is an infinite number of them.

Combine these to get one statement:

A is invertible iff $A\vec{x}=\vec{0}$ has a unique solution.

Q: How to find A^{-1} for an invertible A?

A: Really it is just a matter of taking the system $A\vec{x}=\vec{y}$ and solving for $\vec{x}$ as a function of $\vec{y}$. Then the coefficients of $\vec{y}$ are A^{-1}.

That is, if you can take the system

(✳) $A\vec{x}=\vec{y}$, or
$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \qquad \ddots \qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = y_n \end{matrix}$$

and slowly turn it into the system

(★) $\vec{x}=B\vec{y}$, where
$$\begin{matrix} x_1 = b_{11}y_1 + \cdots + b_{1n}y_n \\ \vdots \qquad \ddots \qquad \vdots \\ x_n = b_{n1}y_1 + \cdots + b_{nn}y_n \end{matrix}$$

Then you "found" a way of going back from $\vec{y}$ to $\vec{x}$ via a linear transf.

This is precisely what the inverse lin. transf does.

In matrix form, this means:

1. Create a new $n\times 2n$ matrix $[A_{n\times n} \,\vdots\, I_n]$
2. Compute $\mathrm{rref}[A \,\vdots\, I_n]$

At the end, you will get a matrix of the form $[I_n \,\vdots\, B_{n\times n}]$ (if A is invertible).

Then $B=A^{-1}$, and is the coefficient matrix of (★).