Class 5: 9/13/13

Linear transformations $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$
can only behave in certain limited ways.

Understanding the different ways is ke to understanding
linear transformations.

For $n = m = 2$, all lin. transf. are combinations
(compositions) of these types:

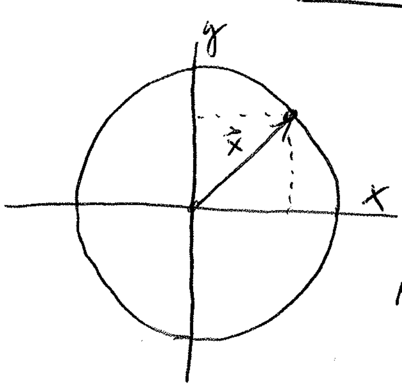
(A) <u>Scalings</u>

When $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, for $k \in \mathbb{R}$,

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}, \text{ or } T(\vec{x}) = k\vec{x}$$

- Here, if $k > 0$, direction is preserved $\forall \vec{x} \in \mathbb{R}^2$, and every vector is scaled by k
- If $k < 0$, direction is reversed (but again scaled)
- What happens when $k = 0$?

(B) <u>Rotations</u>



Any vector $\vec{x} \in \mathbb{R}^2$ has polar coordinates
$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \text{ for some } 0 \le \theta < 2\pi, \; r \geq 0$$

For pts in $\mathbb{R}^2$ <u>ON</u> the unit circle, where $x^2 + y^2 = 1$
$r = 1$, and $\vec{x} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $0 \le \theta \le 2\pi$

Ⓑ Rotations (cont'd)

Given the 2 standard vectors $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, we say they are <u>perpendicular</u>, or <u>orthogonal</u> since they form a $90°$ ($\frac{\pi}{2}$ radians) angle.

In general, 2 vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are called orthogonal if their dot product is 0.

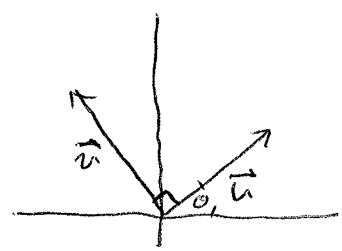
Note: the dot product is <u>defined</u> for 2 column vectors by rewriting the first as a row vector and using regular matrix multiplication.

$$\vec{x} \cdot \vec{y} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \cdot \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} = [x_1 \cdots x_n]\begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} = \sum_{i=1}^{n} x_i y_i, \text{ a scalar.}$$

Also note that the 2 vectors, for any $\theta \in [0, 2\pi)$

$$\vec{u} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

are <u>orthogonal</u>.



Consider the linear transformation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$T\vec{e}_1 = T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}, \quad T\vec{e}_2 = T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

for a choice of $\theta \in [0, 2\pi)$

The matrix A_θ for this transformation can be written

$$T_\theta(\vec{x}) = A_\theta \vec{x}, \text{ where } A_\theta = \begin{bmatrix} | & | \\ T\vec{e}_1 & T\vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

(B) Rotations (cont'd).

A_θ here is called a (counterclockwise) rotation of $\mathbb{R}^2$ through an angle θ.

<u>See the pattern?</u>

A linear transformation on $\mathbb{R}^2$ with a matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \text{ where } a^2+b^2=1$$

is a <u>rotation</u> of the plane.

ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix}$ is a rotation of angle $\pi/3$ (solution to $1/2 = \cos\theta$, $\frac{\sqrt{3}}{2} = \sin\theta$)

ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation in $\mathbb{R}^2$

Through what angle?

Q: What if $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, but $a^2+b^2 \neq 1$?

Is this a rotation?

A: Using polar coordinates, $a = r\cos\theta$, $b = r\sin\theta$

we see that

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= rI_2\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Hence $T(\vec{x}) = A\vec{x}$ corresponds to a composition of a rotation and a <u>scaling</u>.

(c) Shear

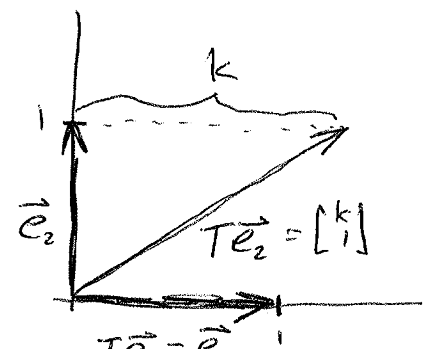
A matrix of the form $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ corr to a linear transformation called a horizontal <u>shear</u> (vertical if $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$).

Notice that $T(\vec{x}) = T\begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix}$.
So what does it do to points in the plane?

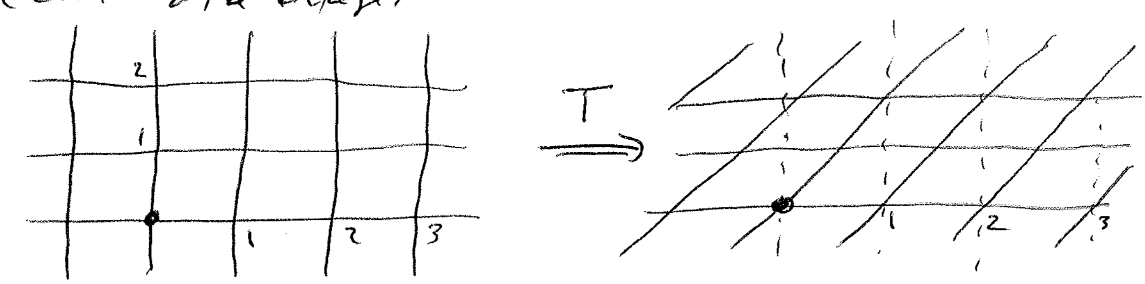
Geometrically

$$T\vec{e_1} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e_1}$$

$$T\vec{e_2} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}.$$



Think of all integer lines in the plane as a grid
(~~called the integer~~

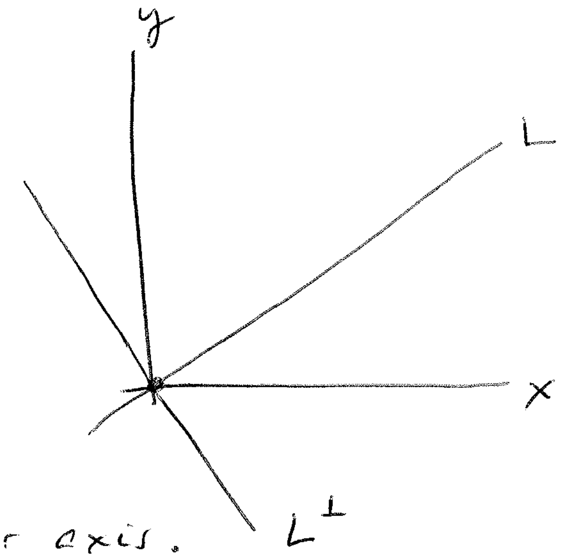


Via T, the grid goes to a "slanted" (in the vertical direction) grid.

ex $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(\vec{x}) = A\vec{x}$
takes all horizontal lines to themselves
and all vertical lines to diagonal lines
(of slope 1)

(D) Projections

Let L be a line thru the origin in $\mathbb{R}^2$. L has a unique perpendicular line thru the origin, $L^\perp$



(for L on axis, $L^\perp$ is the other axis. Otherwise slope of $L^\perp$ is $-\frac{1}{m}$ for L w/ slope m)

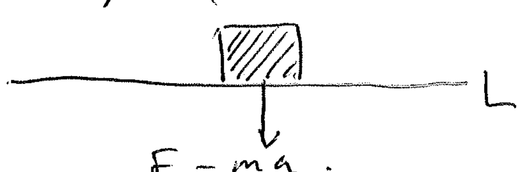
Any vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as a sum of a vector along L and a vector along $L^\perp$. Call those $\vec{x}^{\,\parallel}$, $\vec{x}^{\,\perp}$, respectively.

ex. If L = x-axis, then for $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$,

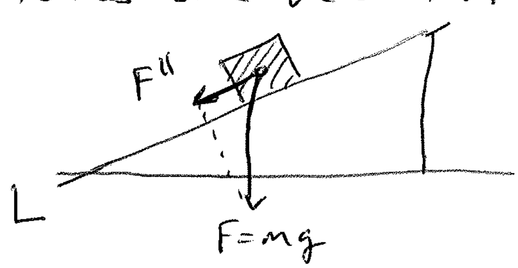
$$\vec{x}^{\,\parallel} = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ and } \vec{x}^{\,\perp} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Q: How to find these component vectors if L is not on axis?

Note: Why do this? Many reasons. For example in classical mechanics, forces are vectors.

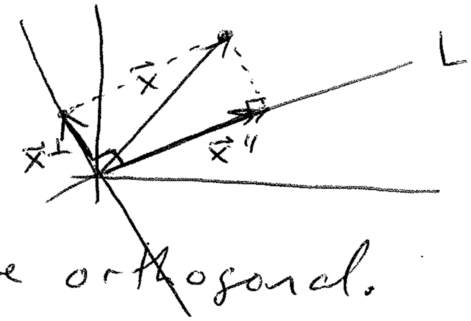


Motion restricted to along L. Block doesn't move since no component of F along L.



Here there is a component of F along L, $F^\parallel$. Will block move?

For general L, and $\vec{x}$ one can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

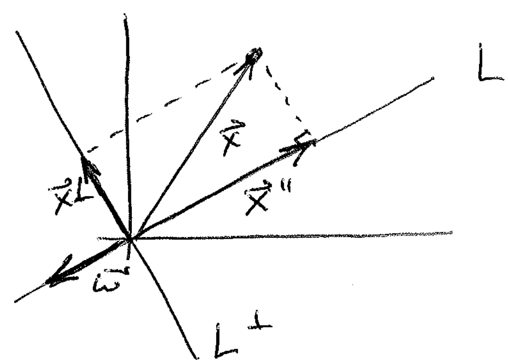


Notice that $\vec{x}^{\parallel} \cdot \vec{x}^{\perp} = 0$

Here $\vec{x}^{\parallel} \perp \vec{x}^{\perp}$ or they are orthogonal.

Definition Given L, the lin. transf. $T_L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_L(\vec{x}) = \vec{x}^{\parallel}$ is called the orthogonal projection of $\vec{x}$ onto L, and denoted $\mathrm{proj}_L(\vec{x}) = \vec{x}^{\parallel}$.

Q: What does the matrix look like here? How to construct it?



A: Choose any nonzero $\vec{\omega} \in L$.

Then $\exists k \in \mathbb{R}$ where

$$k\vec{\omega} = \vec{x}^{\parallel}$$

How to calculate the k? Here $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$
$= \vec{x} - k\vec{\omega}$

and since $\vec{x}^{\parallel}$, $\vec{x}^{\perp}$ are orthogonal,

$$\underbrace{(\vec{x} - k\vec{\omega})}_{\vec{x}^{\perp}} \cdot \underbrace{\vec{\omega}}_{\frac{1}{k}\cdot\vec{x}^{\parallel}} = 0$$

This is $\vec{x} \cdot \vec{\omega} - k\,\vec{\omega} \cdot \vec{\omega} = 0$, or $k = \dfrac{\vec{x} \cdot \vec{\omega}}{\vec{\omega} \cdot \vec{\omega}}$

So that $\mathrm{proj}_L(\vec{x}) = \vec{x}^{\parallel} = k\vec{\omega} = \underbrace{\left(\dfrac{\vec{x} \cdot \vec{\omega}}{\vec{\omega} \cdot \vec{\omega}}\right)}_{\text{just a scalar. } k} \vec{\omega}$

(D) Projections (cont'd)

And if $\vec{w}$ is chosen to be a unit vector $\vec{u}$ (a vector of length 1), then $(\vec{u}\cdot\vec{u}=1)$, and

$$(*) \qquad \text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = \underbrace{(\vec{x}\cdot\vec{u})}_{k.}\,\vec{u}$$

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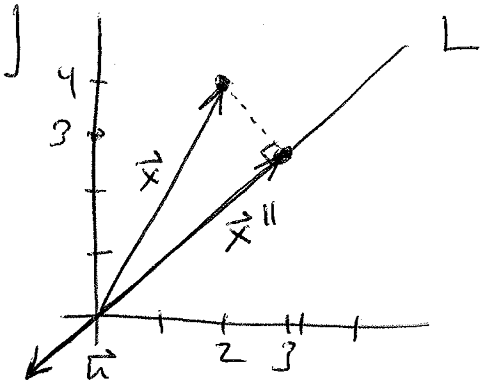
ex. Calculate $\vec{x}^{\parallel}$ for $\vec{x}=\begin{bmatrix}2\\4\end{bmatrix}$ and $L=\{y=x \text{ line}\}$.

<u>strategy</u>: choose $\vec{u}$ of length 1 and entries equal, and use formula $(*)$.

<u>solution</u>: Choose $\vec{u}=\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

Here $\|\vec{u}\|$ = length of $\vec{u}$

$$=\sqrt{\left(-\tfrac{1}{\sqrt{2}}\right)^2+\left(-\tfrac{1}{\sqrt{2}}\right)^2}=1$$



and.

$$\text{proj}_L\left(\begin{bmatrix}2\\4\end{bmatrix}\right)=\left(\begin{bmatrix}2\\4\end{bmatrix}\cdot\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\right)\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$=\left(2\left(-\tfrac{1}{\sqrt{2}}\right)+4\left(-\tfrac{1}{\sqrt{2}}\right)\right)\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$=-\frac{6}{\sqrt{2}}\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}.$$

Ⓓ Projections (cont'd.)

So what is the matrix of the lin. transf.?

$$\begin{aligned} \mathrm{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} &= \left(\begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= (xu_1 + yu_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \\ &= \begin{bmatrix} xu_1^2 + yu_1u_2 \\ xu_1u_2 + yu_2^2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \\ &= A\vec{x}. \end{aligned}$$

where $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $u_1^2 + u_2^2 = 1$ $(\|\vec{u}\| = 1)$.

ex: ~~The matrix~~ Here $T(\vec{x}) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\vec{x}$ is the orthogonal projection onto the line L containing the vector $\vec{u} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

ex: Given the orthogonal projection onto $L = \{y = x\}$, calculate $\vec{x}^{\parallel}$ for $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Solution: Here $T(\vec{x}) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\vec{x}$.

So $T\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(\frac{1}{2}) + 4(\frac{1}{2}) \\ 2(\frac{1}{2}) + 4(\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Q: What does A look like for $\vec{\omega}$?

(E) Reflections

In the same way, the linear transformation of $\mathbb{R}^2$ which is a reflection of $\mathbb{R}^2$ through a line L through the origin is simply:

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} \quad \text{(why is this so?)}$$

Recall $\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}$ and $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$
$= \vec{x} - \text{proj}_L(\vec{x})$.

Hence
$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = \text{proj}_L(\vec{x}) - (\vec{x} - \text{proj}_L(\vec{x}))$$
$$= 2\,\text{proj}_L(\vec{x}) - \vec{x}$$
$$= 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x} \quad \text{for } \vec{u} \in L \text{ of length } 1.$$

Write this out to see that for $T: \mathbb{R}^2 \to \mathbb{R}^2$ where

$$T(\vec{x}) = \text{ref}_L(\vec{x}) = A\vec{x}, \text{ where } A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}\vec{x}, \quad a^2+b^2=1.$$

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ex. Let $T(\vec{x}) = A\vec{x} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$. Then T is a reflection about what line?

- Trig helps here a lot.
- Can you see the line of reflection?