Class 4: 9/11/13

Q1 Is $T_1 : \mathbb{R} \to \mathbb{R}$ $T_1(x) = 3x$ a linear transf?

Q2: Is $T_2 : \mathbb{R} \to \mathbb{R}$ $T_2(x) = x^2$?

Q3: Is $T_3 : \mathbb{R} \to \mathbb{R}$ $T_3(x) = 3x + 2$?

Q4: How does one recognize a linear ~~transf~~ equation?

a linear transf?

---

2 useful constructions

(I) Just like functions, 1-1 linear transformations have inverses functions from their range back to their domain.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be 1-1 and linear
(it will be onto in this case).

Then the inverse relation will exist and be a function also. Call it $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$

If $T(\vec{x}) = A\vec{x}$, then T^{-1} will be linear, and
$T^{-1}(\vec{x}) = B\vec{x}$. Here, B is the inverse matrix of A, in the sense that

(*) $$(T \circ T^{-1})(\vec{x}) = A(B\vec{x}) = AB\vec{x} = I_n\vec{x} = \vec{x} = BA\vec{x} = B(A\vec{x}) = (T^{-1} \circ T)(\vec{x})$$

ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$.
Then $T^{-1}(\vec{x}) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\vec{x}$

① Check the parts of (*) above.

② On vectors: Choose $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$, $T^{-1}\begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

II Constructing A for a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$.

ⓐ If T is given by a system of linear eqns, A is simply the coefficient matrix:

ex. $$\left.\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 \\ y_2 &= 4x_4 + 5x_5 + 3x_6 \\ y_3 &= -2x_1 + x_3 \end{aligned}\right\} \quad \vec{y} = A\vec{x}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ -2 & 0 & 1 \end{bmatrix}.$$

ⓑ Sometimes, the matrix A is hidden in the vector:

ex. Find A for the lin. transf. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xmapsto{A} \begin{bmatrix} x_2 \\ 3x_1 - x_2 \end{bmatrix}$

Here RHS is $$\begin{bmatrix} x_2 \\ 3x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 \\ 3x_1 - 1x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}}_{A.} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(Also written $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 3x_1 - x_2 \end{bmatrix}$.)

ⓒ Sometimes, you are told T is linear but only given a few function assignments.

Can you "construct" A?

II With a linear transformation,

In $\mathbb{R}^n$, the vector $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ith place is called the ith <u>standard vector</u> in $\mathbb{R}^n$

Given $A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, we have

$$A\vec{e}_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ the ith column of } A$$ (see box pg 48)

So that for $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$, we get

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}.$$

We can use this to "construct" A for a linear transformation when we know certain values only:

ex. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, where we only know

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Find $A_{2\times 2}$ so that $T(\vec{x}) = A\vec{x}$.

<u>Strategy</u>: Use linearity to find how T acts on $\vec{e}_1$ and $\vec{e}_2$. Then construct A.

Solution: Here we know

$$① \begin{bmatrix} 3 \\ 2 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \overset{\text{Linearity}}{=} T\vec{e}_1 + T\vec{e}_2$$

$$② \begin{bmatrix} -2 \\ 1 \end{bmatrix} = T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\vec{e}_1 - 2T\vec{e}_2$$

These are both vector equations in 2 unknown vectors $T\vec{e}_1$ and $T\vec{e}_2$

We can solve for them in the same way we solved systems in Chapter 1.

First, equation ① − equation 2.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = T\vec{e}_1 + T\vec{e}_2 - (T\vec{e}_1 - 2T\vec{e}_2)$$

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 3T\vec{e}_2 \Rightarrow \boxed{\begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix} = T\vec{e}_2}$$

And 2 equation ① + equation ②

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2(T\vec{e}_1 + T_2\vec{e}_2) + T\vec{e}_1 - 2T\vec{e}_2$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 3T\vec{e}_1 \Rightarrow \boxed{\begin{bmatrix} 4/3 \\ 5/3 \end{bmatrix} = T\vec{e}_1}$$

Here, $T(\vec{x}) = A\vec{x}$, when $A = \begin{bmatrix} T\vec{e}_1 & T\vec{e}_2 \end{bmatrix} = \begin{bmatrix} 4/3 & 5/3 \\ 5/3 & 1/3 \end{bmatrix}$.

Does it work?

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 & 5/3 \\ 5/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/3 \\ 6/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \checkmark$$

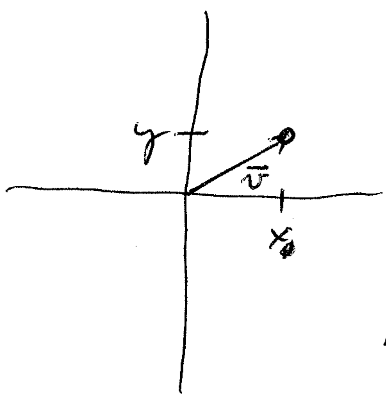
$$T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4/3 & 5/3 \\ 5/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4/3 - 10/3 \\ 5/3 - 2/3 \end{bmatrix} = \begin{bmatrix} -6/3 \\ 3/3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark$$

Geometric Interpretations of linear transformations.

Some examples

linear transformations take vectors to vectors in particularly nice ways. Here we will show how the form of $A_{2\times2}$ affects the function (LT) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = A\vec{x}$.

__Question__: Describe where vectors "go" under the LT $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T_3(\vec{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$, ~~[illegible]~~.



Here for any $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $T(\vec{v}) = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$.

every vector is scaled by 3. and stays in the same direction

in fact, for any $k > 0$, $T_k(\vec{v}) = k\vec{v}$ scales $\vec{v}$ by mult by k. (same direction).

What happens when $k < 0$? When $k = 0$?

This kind of linear transformation is called a scaling.

- direction pres. if $k > 0$
- direction reversing if $k < 0$
- dilation if $|k| > 1$,
- contraction if $0 < |k| < 1$.