Class 3: 9/9/13

Denote a matrix with its entries

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nm} \end{bmatrix}$$

By convention, (a) if $m=1$, called a (column) n-vector

(b) if $n=1$, called a row m-vector

(c) if $m=n=1$, called a scalar and treated as simply a real number

Note: All unspecified vectors are considered column vectors.

Like eqns, matrices have algebraic properties:

(I) They can be added together iff they are of the same size:

$$A_{n\times m} + B_{n\times m} = C_{n\times m}, \quad a_{ij} + b_{ij} = c_{ij}$$

(II) They can be multiplied together iff their dimensions are compatible:

$$A_{n\times m} \cdot B_{p\times r} = C_{n\times r} \quad \text{iff } m=p$$

(II) cont'd. How does one multiply matrices?

Def The dot product of a row n-vector and a ~~row~~ column n-vector is defined

$$\vec{w} \cdot \vec{v} = [w_1 \cdots w_n] \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} w_i v_i = \text{a scaler, a real \#.}$$

Hence the matrix product above defined $C_{n \times r}$

with $c_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$,
$1 \le i \le n \qquad 1 \le j \le r$

(III) Any matrix can be multiplied by a scaler, and

$$k A_{n \times m} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & \ddots & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

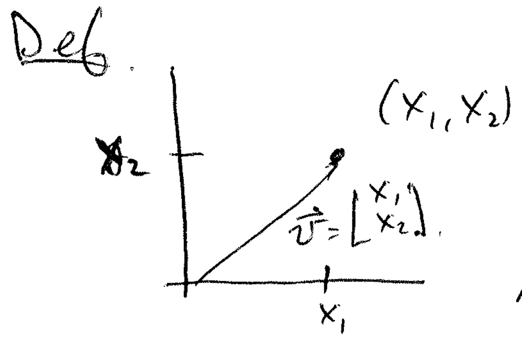
(IV) Addition and multiplication of matrices satisfy.

i) $A(B+C) = AB + AC$

ii) $k(AB) = A(kB)$

at least when the operations make sense.

Def.



The set of all ordered n-tuples of numbers $(x_1, \cdots, x_n) \in \mathbb{R}^n$ identifies with the set of all (column) n-vectors. $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{v}$

Hence $\mathbb{R}^n = \left\{ \vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \in \mathbb{R} \right\}$

Back to our system of equations in
matrix form from last week:

$$A\vec{x} = \vec{b}$$

In its most general form

$$A_{n\times m}\vec{x}_{m\times 1} = \vec{b}_{n\times 1}$$

gives us a good interpretation of a matrix $A_{n\times m}$ beyond the simple bookkeeping tool for linear equations: <u>It</u> <u>is</u> <u>a</u> <u>function</u>, whose input is an m-vector and whose output is an n-vector.

Define $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ as the fnc $T(\vec{x}) = A\vec{x} = \vec{b}$

Note: We also write sometimes $\vec{x} \xmapsto{A} \vec{b}$
or $\vec{x} \xmapsto{T} \vec{b}$

<u>Def</u>. A function $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a <u>linear transformation</u> if $\exists\ A_{n\times m}$ where $T(\vec{x}) = A\vec{x}$

Back in Chapter 1, and systems of eqns.

(a) If $\vec{x}$ is specified but not $\vec{b}$, then calculate $\vec{b}$ via matrix multiplication.

(b) If $\vec{b}$ is specified but not $\vec{x}$, then one must solve the system $A\vec{x} = \vec{b}$ if possible

(c) $A_{n \times m}$ is simply the coefficient matrix.

Special linear transformations

- $m = n$: The matrix is called square, and can have special properties.
  ex: As a function, it may have an <u>inverse</u>.

- $m = 1$: T specifies a (parameterized) line in $\mathbb{R}^n$;

- When $A_{n \times m} = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ The identity matrix

Here $I_n \vec{x} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$

for all or for every.

If a function $T: \mathbb{R}^m \to \mathbb{R}^n$ is specified by a matrix, so that $T(\vec{x}) = A\vec{x}$, then T is a linear transformation.

If you are not sure whether such a matrix exists, T will be linear iff it satisfies

ⓐ $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m$

ⓑ $T(k\vec{v}) = kT(\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^m, k \in \mathbb{R}$

Caution: The plus signs in ⓐ are different: on the left is addition in $\mathbb{R}^m$. On the right is addition in $\mathbb{R}^n$.

Here ⓐ and ⓑ determine "linearity" of T.

When T is linear, (satisfies ⓐ and ⓑ) then a matrix A will exist so that $T(\vec{x}) = A\vec{x}$.

2 useful constructions

(I) Just like functions, 1-1 linear transformations have inverses functions from their range back to their domain.

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be 1-1 and linear (it will be onto in this case).

Then the inverse relation will exist and be a function also. Call it $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$

If $T(\vec{x}) = A\vec{x}$, then T^{-1} will be linear, and $T^{-1}(\vec{x}) = B\vec{x}$. Here, B is the inverse matrix of A, in the sense that

(*) $$(T \circ T^{-1})(\vec{x}) = A(B\vec{x}) = AB\vec{x} = I_n\vec{x} = \vec{x} = BA\vec{x} = B(A\vec{x}) = (T^{-1} \circ T)(\vec{x})$$

ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$.

Then $T^{-1}(\vec{x}) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\vec{x}$

(1) Check the parts of (*) above.

(2) On vectors: Choose $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \quad T^{-1}\begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(II) Constructing A

In $\mathbb{R}^n$, the vector $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ith place is called the <u>ith</u> <u>standard</u> <u>vector</u> in $\mathbb{R}^n$

Given $A_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, we have

$$A\vec{e}_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ the ith column of } A \quad \begin{pmatrix} \text{see box} \\ \text{pg 48} \end{pmatrix}$$

So that for $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$, we get

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}.$$

We can use this to "construct" A for a linear transformation when we know certain values only:

ex. Given $T: \mathbb{R}^2 \to \mathbb{R}^2$, where we only know

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Find $A_{2\times 2}$ so that $T(\vec{x}) = A\vec{x}$.

<u>Strategy</u>: Use linearity to find how T acts on $\vec{e}_1$ and $\vec{e}_2$. Then construct A.

Solution: Here we know

$$① \begin{bmatrix}3\\2\end{bmatrix} = T\begin{bmatrix}1\\1\end{bmatrix} = T\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right) \overset{\text{Linearity}}{=} T\vec{e}_1 + T\vec{e}_2$$

$$② \begin{bmatrix}-2\\1\end{bmatrix} = T\begin{bmatrix}1\\-2\end{bmatrix} = T\left(\begin{bmatrix}1\\0\end{bmatrix}-2\begin{bmatrix}0\\1\end{bmatrix}\right) = T\vec{e}_1 - 2T\vec{e}_2$$

These are both vector equations in 2 unknown vectors $T\vec{e}_1$ and $T\vec{e}_2$

We can solve for them in the same way we solved systems in Chapter 1.

First, <u>equation ① − equation 2.</u>

$$\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} = T\vec{e}_1 + T\vec{e}_2 - (T\vec{e}_1 - 2T\vec{e}_2)$$

$$\begin{bmatrix}5\\1\end{bmatrix} = 3T\vec{e}_2 \implies \boxed{\begin{bmatrix}5/3\\1/3\end{bmatrix} = T\vec{e}_2}$$

And <u>2 equation ① + equation ②</u>

$$\begin{bmatrix}4\\5\end{bmatrix} = 2\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} = 2(T\vec{e}_1 + T_2\vec{e}_2) + T\vec{e}_1 - 2T\vec{e}_2$$

$$\begin{bmatrix}4\\5\end{bmatrix} = 3T\vec{e}_1 \implies \boxed{\begin{bmatrix}4/3\\5/3\end{bmatrix} = T\vec{e}_1}$$

Here, $T(\vec{x}) = A\vec{x}$, when $A = \begin{bmatrix}T\vec{e}_1 & T\vec{e}_2\end{bmatrix} = \begin{bmatrix}4/3 & 5/3\\5/3 & 1/3\end{bmatrix}$.

Does it work?

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4/3 & 5/3\\5/3 & 1/3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}9/3\\6/3\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} \checkmark$$

$$T\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}4/3 & 5/3\\5/3 & 1/3\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}4/3-10/3\\5/3-2/3\end{bmatrix} = \begin{bmatrix}-6/3\\3/3\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix} \checkmark$$

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