Class 24: Oct 30, 2013

There are a bunch of very useful properties regarding Ⓐ Orthogonal matrices
Ⓑ Transposes

Go over them in detail.

<u>Last big Thm in 5.3</u>

Let $V \subset \mathbb{R}^n$ be a m-dim. subspace with an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$, and $T: \mathbb{R}^n \to \mathbb{R}^n$ an orthogonal projection onto V, so that $\text{im}(T) = V$.

Then the matrix for T is the matrix $P_{n\times n}$ where

$P_{n\times n} = QQ^T$, where $Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$ is $n \times m$.

<u>Notes</u> ① Here, order is very important, as Q^TQ is of size $m \times m$ and cannot be the matrix of a $T : \mathbb{R}^n \to \mathbb{R}^n$

② P is a symmetric matrix. Recall that our orthogonal projections of Chapter 2 were symmetric: $T : \mathbb{R}^2 \to \mathbb{R}^2$ projecting down to a line L orthogonally had form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ for $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ a unit vector in L.

③ Once you have an orthonormal basis $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_m\}$ for $V \subset \mathbb{R}^n$ a subspace, you can easily calculate its effect on vectors:

$$T(\vec{x}) = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

Now you can also easily calculate its matrix P as above.

ex. Recall from the first Midterm:

$T(\vec{x}) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \vec{x}$ is a composition of a scaling and an orthogonal projection onto a line $L = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$. Determine the orthogonal projection part.

Strategy: Use the previous thm. directly.

Solution: Here, we can view the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ as a basis for $L \subset \mathbb{R}^2$, so $\mathcal{B} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$. It is not orthonormal, though. Following Gram-Schmidt, we produce $\mathcal{C} = \{\vec{u}\}$, where

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}, \text{ for } \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$\mathcal{C}$ here is an orthonormal basis for L.

Here we form $Q = \begin{bmatrix} | \\ \vec{u}_1 \\ | \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$.

By the theorem, we can now form P so that P is the orthogonal projection of $\mathbb{R}^2$ to L

or $$P_{2\times 2} = Q_{2\times 1} \cdot Q^T_{1\times 2} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$
$$= \begin{bmatrix} 4/\sqrt{5} & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

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notes

① If you recall, on the exam, we computed

$$T(\vec{x}) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} \vec{x}$$

where the latter was the orthogonal projection.

② Think of the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ as the matrix $M_{2\times 1}$ whose columns are lin. indep. Then $M_{2\times 1}$ has a QR factorization into $Q_{2\times 1}$ whose column is orthonormal, and $R_{1\times 1}$ which is the change of basis scalar. Here

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = M_{2\times 1} = Q_{2\times 1} R_{1\times 1} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5} \end{bmatrix}.$$

$\boxed{\text{This is our } Q \text{ above}}$

Section 5.5

We will skip ~~Section~~ Section 5.4 and also the latter part of Section 5.5 on Fourier Analysis.

We will cover the inner product stuff, though.

The dot product, $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i = \vec{x}^T \vec{y}$, defined for vectors in $\mathbb{R}^n$, is an example of a <u>scalar product</u>: A multiplication of 2 elements of a linear space whose result is a number (a scalar).

There are other examples, and this section will work with them.

Def An inner product (scalar product) in a linear space V is a rule assigning a number (a scalar) to any pair of elements of V, (denoted $\langle f, g \rangle$, for $f, g \in V$), so that:

(A) $\langle f, g \rangle = \langle g, f \rangle$ symmetric

(B) $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

(C) $\langle cf, g \rangle = c\langle f, g \rangle$

acts linearly on combinations

(D) $\langle f, f \rangle \geq 0$ always, and $\langle f, f \rangle = 0$ only when f is the neutral element in V. Called positive definite.

Notes ① Due to Ⓑ and Ⓒ above, we know that the transformation $T_g : V \to \mathbb{R}$

$$T_g(f) = \langle f, g \rangle$$

for some fixed $g \in V$, is a linear transformation, while the transformation

$T : V \to \mathbb{R}$, $T(f) = \langle f, f \rangle$ is not

($T(cf) = \langle cf, cf \rangle = c^2 \langle f, f \rangle$ and not equal to $c\langle f, f \rangle$ like it should).

② The dot product on vectors in $\mathbb{R}^n$ is an inner product.

③ The name "inner" product comes from the fact that there is also an "outer" product on vectors:

Inner product: $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = [\vec{x}]\begin{bmatrix}\vec{y}\end{bmatrix} =$ scalar (a 1x1 matrix)

Outer product: $xy^T = \begin{bmatrix}\vec{x}\end{bmatrix}[\vec{y}] = \begin{bmatrix} n\times n \\ \text{matrix} \end{bmatrix}$.

ex. Let $I = [a,b] \subset \mathbb{R}$, where $b > a$. Then the linear space of all continuous functions, $C^0(I)$, has an inner product given by

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt .$$

Show it is an inner product.

Solution Show the 4 conditions are satisfied.

(A) $\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt = \int_a^b g(t) f(t)\, dt = \langle g, f \rangle$

since the order of the product in the integrand does not affect the product.

(B) + (C) $\langle c_1 f_1 + c_2 f_2, g \rangle = \int_a^b (c_1 f_1(t) + c_2 f_2(t))\, g(t)\, dt$

This is how integrals behave

$$= c_1 \int_a^b f_1(t) g(t)\, dt + c_2 \int_a^b f_2(t) g(t)\, dt$$

$$= c_1 \langle f_1, g \rangle + c_2 \langle f_2, g \rangle$$

(D) $\langle f, f \rangle = \int_a^b [f(t)]^2\, dt$ ← integrand always nonnegative hence area under curve always nonnegative.

≥ 0

and $\langle f, f \rangle = 0$ only if $f(t) \equiv 0$, the neutral element.