Class 23: Oct 28, 2013

More on the geometry of vectors and lin transf.

Def. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the lengths of all vectors:

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$$

If T is an orthogonal transformation, then its matrix A is called orthogonal.

Caution: There are many notions of orthogonal floating around here. They are all different:

(a) Vectors (2 of them) are orthogonal if dot product is 0.

(b) A projection is orthogonal if it satisfies certain properties.

(c) Above is the def for a linear transf.

(d) A matrix is orthogonal if its lin transf is so.

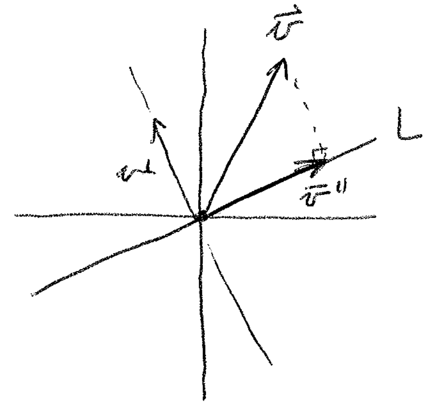
ex. Any rotation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\vec{x}) = A\vec{x}, \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix}, \quad \theta \in [0, 2\pi)$$

is orthogonal. (why?)

ex. Let $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = I_n\vec{x} = \vec{x}$.
T is orthogonal since $\|T(\vec{x})\| = \|I_n\vec{x}\| = \|\vec{x}\|$.

ex. Orthogonal projections are not orthogonal !!!
Why? For $\vec{v}$ not on the line L,



$\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$, and
$\vec{v}^{\perp} \neq \vec{0}$. Hence one can
show $\|\vec{v}\| > \|\vec{v}^{\parallel}\|$.

ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\vec{x}$. Orthogonal?
No, since $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $\left\|\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\| = 2 > 1 = \left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|$.

ex. How about $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$,

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \quad ?$$

Facts ① A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff the vectors $T(\vec{e}_1), \cdots, T(\vec{e}_n)$ form an orthonormal basis for $\mathbb{R}^n$.

Note: Given this T, we construct the matrix A for T as

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

② Any $n \times n$ matrix is orthogonal iff its columns form an orthonormal basis (see last example above!)

③ Let T be an orthogonal lin. transf. If $\vec{u}, \vec{v}$ are orthogonal (so $\vec{u} \cdot \vec{v} = 0$) then $T(\vec{u}) \cdot T(\vec{v}) = 0$

Notes ⓐ Proof is easy. See page 226.

ⓑ Here ③ does not hold for orthogonal projections. $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ may be orthogonal, but their projection onto L will make them lin. dep..

Def. Let $A_{n\times m}$ be a matrix. The <u>transpose</u> of A is a matrix $A^T_{m\times n}$ where the ijth entry of A is the jith entry of A^T (reverse the subscripts).

If $A = A^T$ we call A <u>symmetric</u>

If $A = -A^T$ we call A <u>skew-symmetric</u> or <u>anti-symmetric</u>

ex. For $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{x}^T = [x_1 \cdots x_n]$

ex. For $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A$

A is symmetric

ex. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

ex. Is $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ skew-symmetric. No since the only way $b_{11} = -b_{11}$ is if $b_{11} = 0$.
The main diagonal elements of B must be 0.

Note: The dot product $\vec{x}\cdot\vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$

Using the transpose, we can define dot prod.

as

$$\vec{x}\cdot\vec{y} = \vec{x}^T y = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Big Thm Consider a $n\times n$ matrix A.

A is orthogonal iff $A^T A = I_n$.

or A is orthogonal iff $A^{-1} = A^T$.

pf First statement Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and $A = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}$.

$$\Longrightarrow A^T A = \begin{bmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_n & - \end{bmatrix}\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{u}_1\cdot\vec{u}_1 & \vec{u}_1\cdot\vec{u}_2 & \cdots & \vec{u}_1\cdot\vec{u}_n \\ \vec{u}_2\cdot\vec{u}_1 & \ddots & & \vdots \\ \vdots & & & \vdots \\ \vec{u}_n\cdot\vec{u}_1 & \cdots & \cdots & \vec{u}_n\cdot\vec{u}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = I_n.$$